Нелинейная динамика захваченных течениями поверхностных волн

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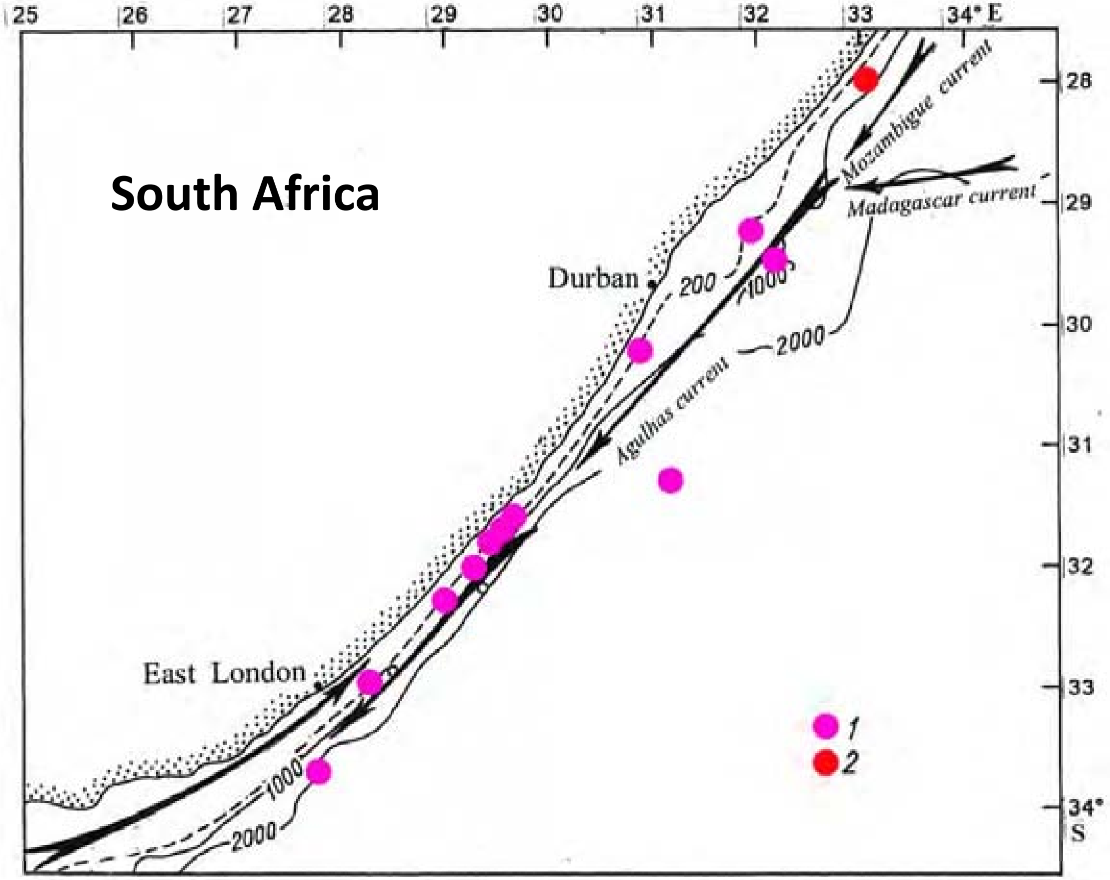
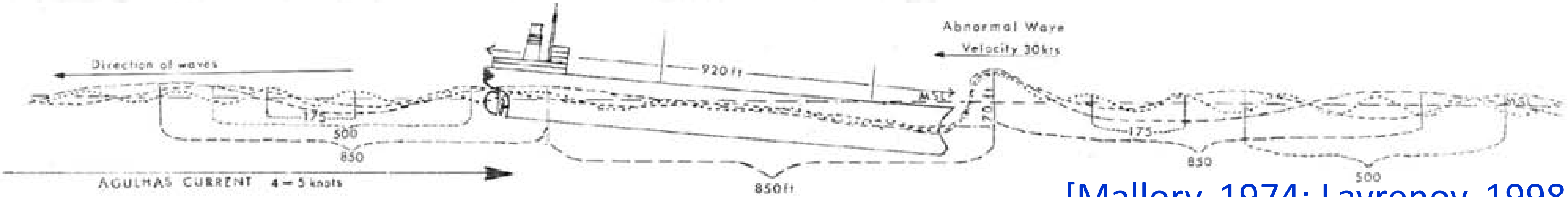
Motivation

Rogue waves on Agulhas current

Suez Canal was closed for 1967-1975, what caused more intense navigation along the coast of Africa.



Gaastekerk (Apr’52)
Oranjefontain (Sep’53)
Jagersfontain (Dec’59)
Edinburgh Castle (Aug’64)
World Glory (Jun’68)
Esso Lancashire (Aug’68)
Clan Maclay (Oct’69)
Southern Cross (Oct’69)
Moreton Bay (Aug’71)
Bencruachan (May’73)
Svealand (Sep’73)
Taganrogsky Zaliv (Apr’85)



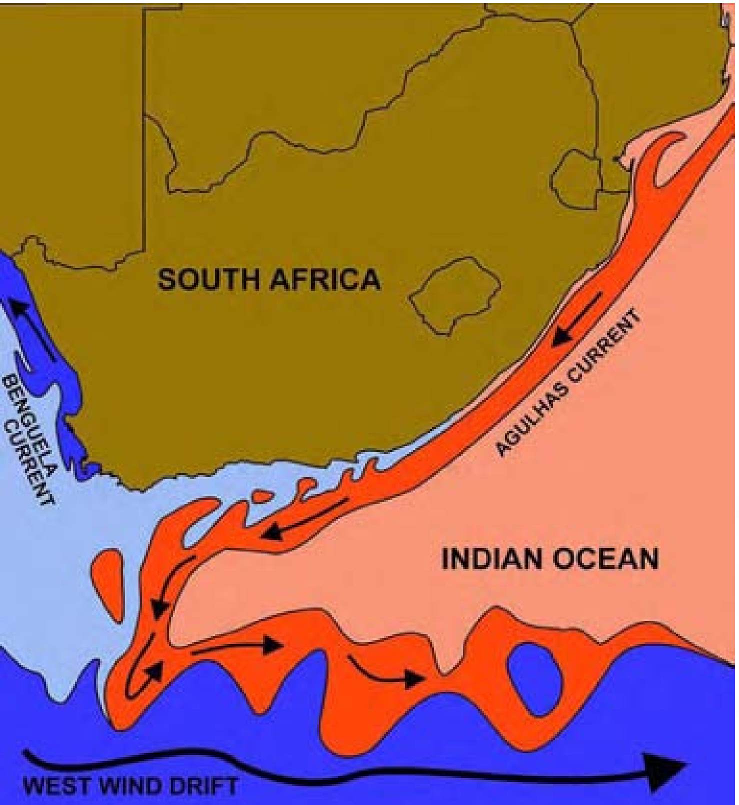
[Mallory, 1974; Lavrenov, 1998]

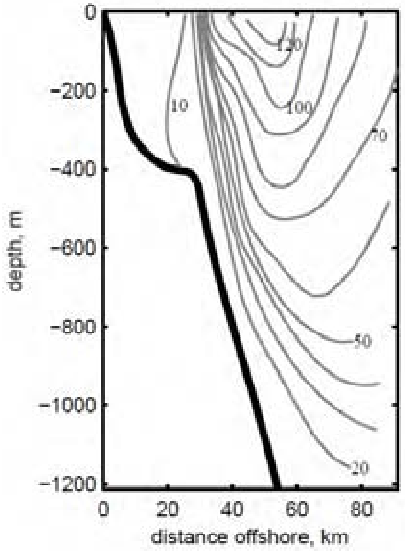
Rogue waves on Agulhas current

12 large accidents with ships in the region of a strong Agulhas current. All accidents occurred close to the maximum of the current

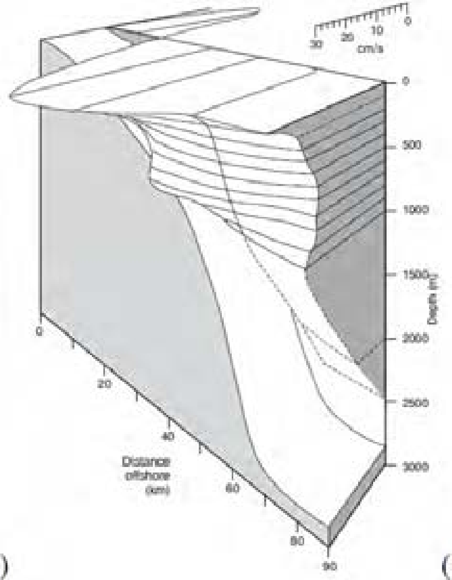
There were coexisting different wave systems.

Waves were propagating against the current.





(a)



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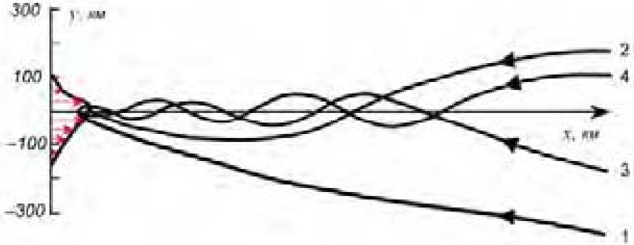
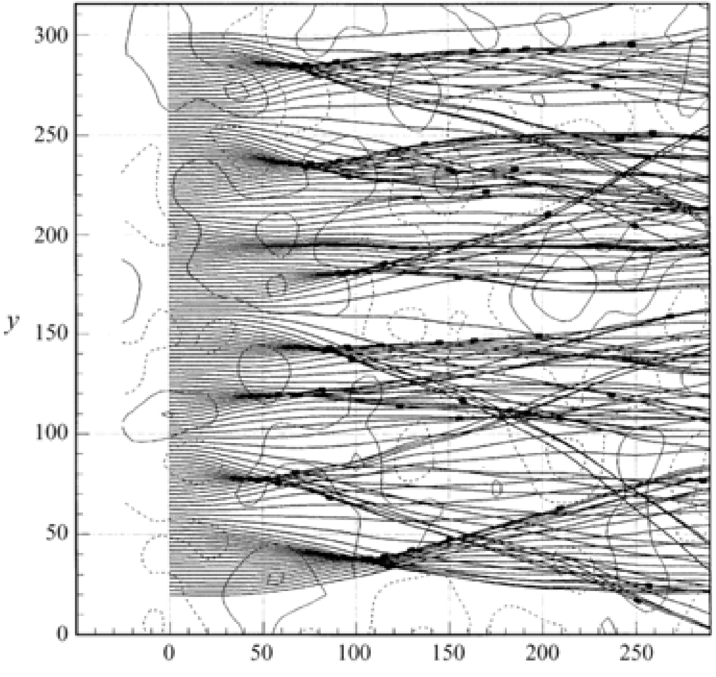
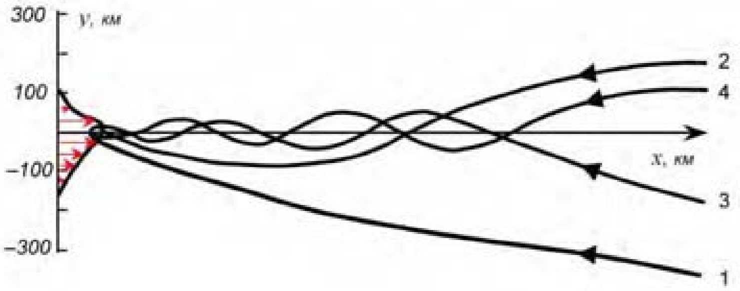


Mechanisms of wave intensification on currents

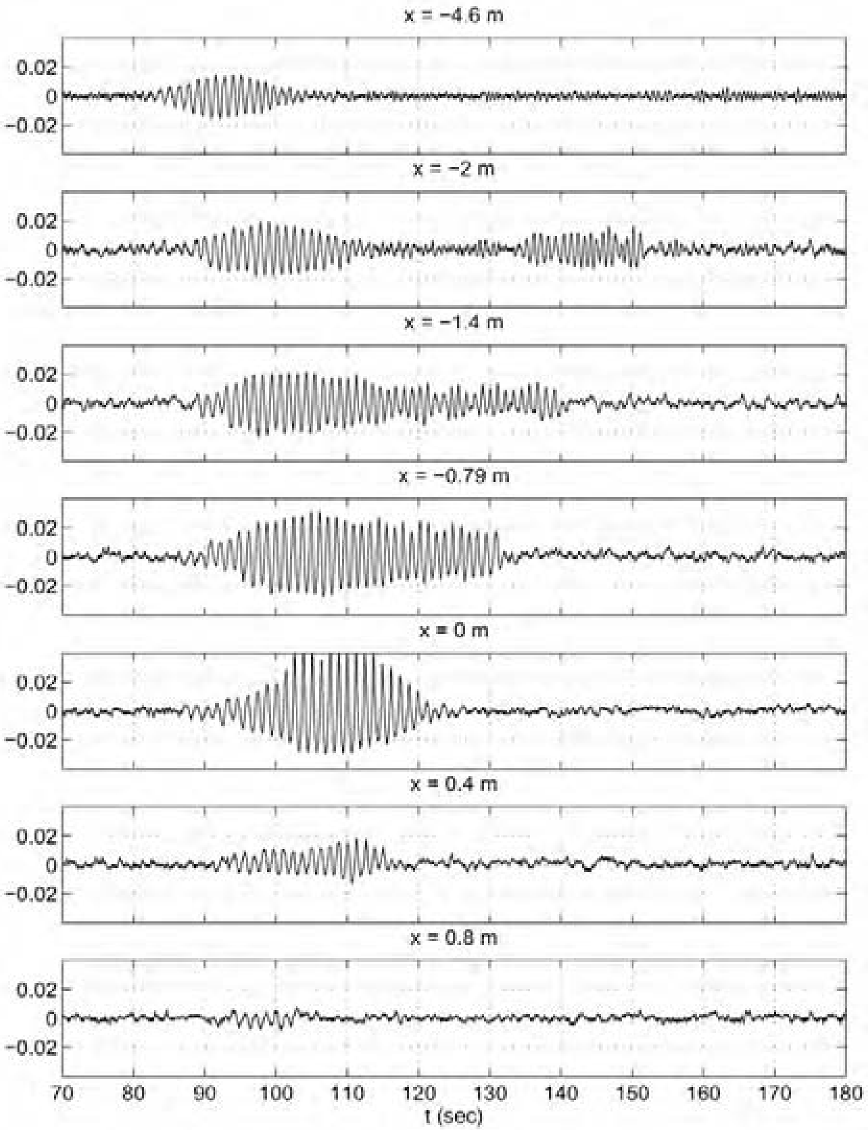
Wave blocking effect, focal zones and caustics, trapped waves. Ray approach



[White & Fornberg, 1998]



[Lavrenov, 1998]



[Chawla & Kirby, 2004]

Mechanisms of wave intensification on currents

Linear models

- Peregrine & Smith (CambPhilSoc’75, RoySoc’79), Peregrine (AdvApplMath’76), Smith (JFM’76): *trapped modes, dispersion relations, ray theory, caustics, nonlinear effects on caustics*
- Lavrenov (NatHaz’98): *rays on a jet current, simulations, Agulhas current conditions*
- White & Fornberg (JFM’98): *statistics for random current fluctuations: a single universal curve*

NLS models: *longitudinally inhomogeneous weak opposite currents*

- Smith JFM1976, Turpin et al, JFM 1983, Gerber JFM 1987, Stocker & Peregrine JFM 1999, Hjelmervik & Trulsen, JFM 2009, Onorato et al, PRL 2011: *increase of wave steepness triggers BF instability and strong departure from Gaussianity*

DNS (strongly nonlinear)

- TT Janssen et al (JFM’06), TT Janssen & Herbers (JPO’09), Moreira & Peregrine (JFM’12): *the increase of steepness on opposing current leads to BF instability, etc.* Non-NLS features: (i) formation of trapped waves. (ii) broad spectrum after the increase of kurtosis.

Observations

- Kudryavtsev et al (JGR 95) observed trapped wind waves on the Gulfstream

Advantage of the modal approach

Conventional theoretical approaches are not adequate for describing nonlinear wave interactions:

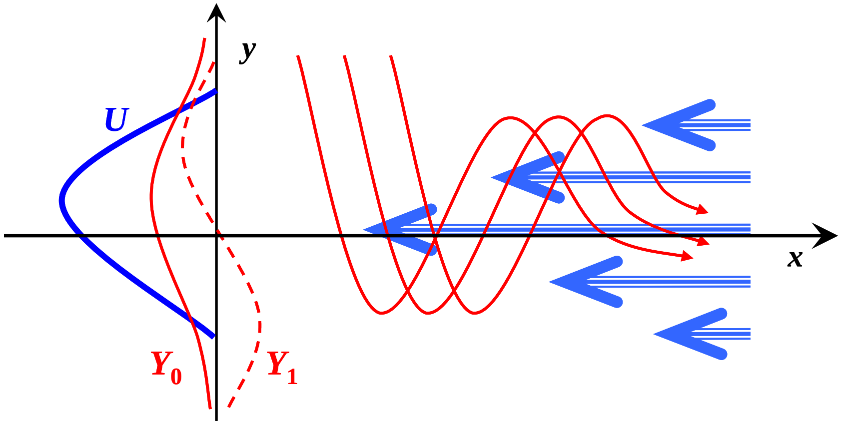
(i) It is implicitly assumed that the nonlinear interactions remain the same and adjust adiabatically, what requires the characteristic scale of nonlinear interactions $L_{nl} > \varepsilon^{-2}\lambda$ ($\varepsilon = ka$ ~< 0.1 is the wave steepness, $\lambda = 2\pi/k$ >~ 100 m is the wave length) be much smaller that the scale of inhomogeneity L_{inh} : $L_{inh} >> L_{nl} > 10$ km.

For the kinetic nonlinear scale $L_{kin} > \varepsilon^{-4}\lambda$ the condition is $L_{inh} >> L_{kin} > 1000$ km.

(ii) Wave refraction occurs in **x**-space, while nonlinear interactions 'live' in the **k**-space.

If the jet current is longitudinally uniform, then the solutions to linearized equations of hydrodynamics for water waves may be always presented in a separable form: as waves propagating along the current with some `modal' dependence on the vertical and transverse variables.

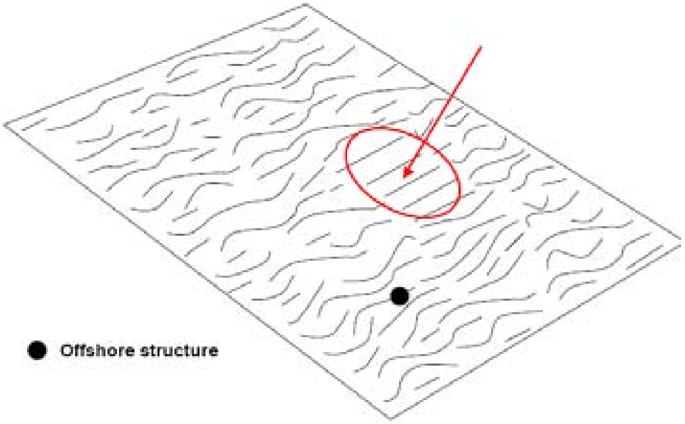
Trapped modes differ qualitatively from the free waves. This fact profoundly changes all aspects of their nonlinear dynamics.



Modulational instability of oceanic waves

There is crucial difference in wave statistics between the 1D evolution and 2D evolution [Onorato et al, PhysFl'02, PRL'09; Waseda, 06; Gramstad & Trulsen, JFM'07, Mori et al, JGR'07].

[Muller et al, 2005]



The weakly nonlinear theory for long wave modulations employs the assumptions of small wave steepness $k_0 a = O(\varepsilon)$, $\varepsilon << 1$ and narrow spectral bandwidth $\Delta k/k_0 = O(\varepsilon)$. The coefficients of the evolution equation are functions of the scaled water depth $k_0 h$.

The nonlinear Schrodinger equation (NLS) for the complex wave envelope $A(x,t)$ takes into account the leading-order terms of nonlinearity and dispersion. In the deep-water limit of the planar geometry it reads

$$i\left(\frac{\partial A}{\partial t} + \frac{\omega_0}{2k_0}\frac{\partial A}{\partial x}\right) + \frac{\omega_0}{8k_0^2}\frac{\partial^2 A}{\partial x^2} + \frac{\omega_0 k_0^2}{2}\left|A\right|^2 A = 0$$

The surface displacement $\eta(x,t)$ can be calculated using the reconstruction formula $\eta(x,t) = \mathrm{Re}[A \exp(i\omega_0 - k_0 t)]$, where $\omega_0 = \omega(k_0)$ according to the dispersion relation.

The NLS equation is integrable by means of the Inverse Scattering Transform (IST). Higher order generalizations of the evolution equation may be derived, which are not integrable.

[Benney & Newell, 1967; Zakharov, 1968; Hasimoto & Ono, 1972; Zakharov & Shabat, 1972]

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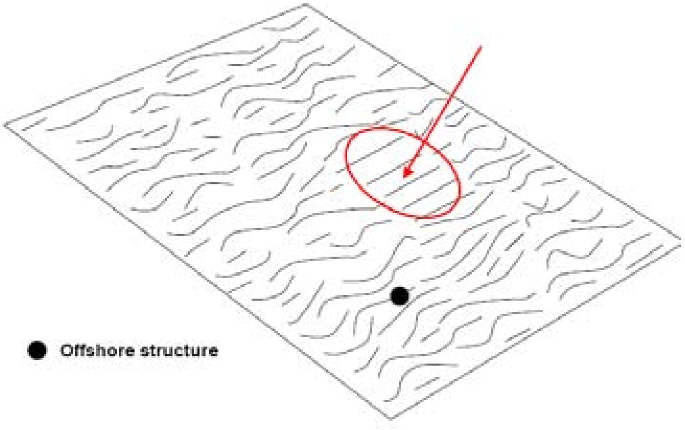
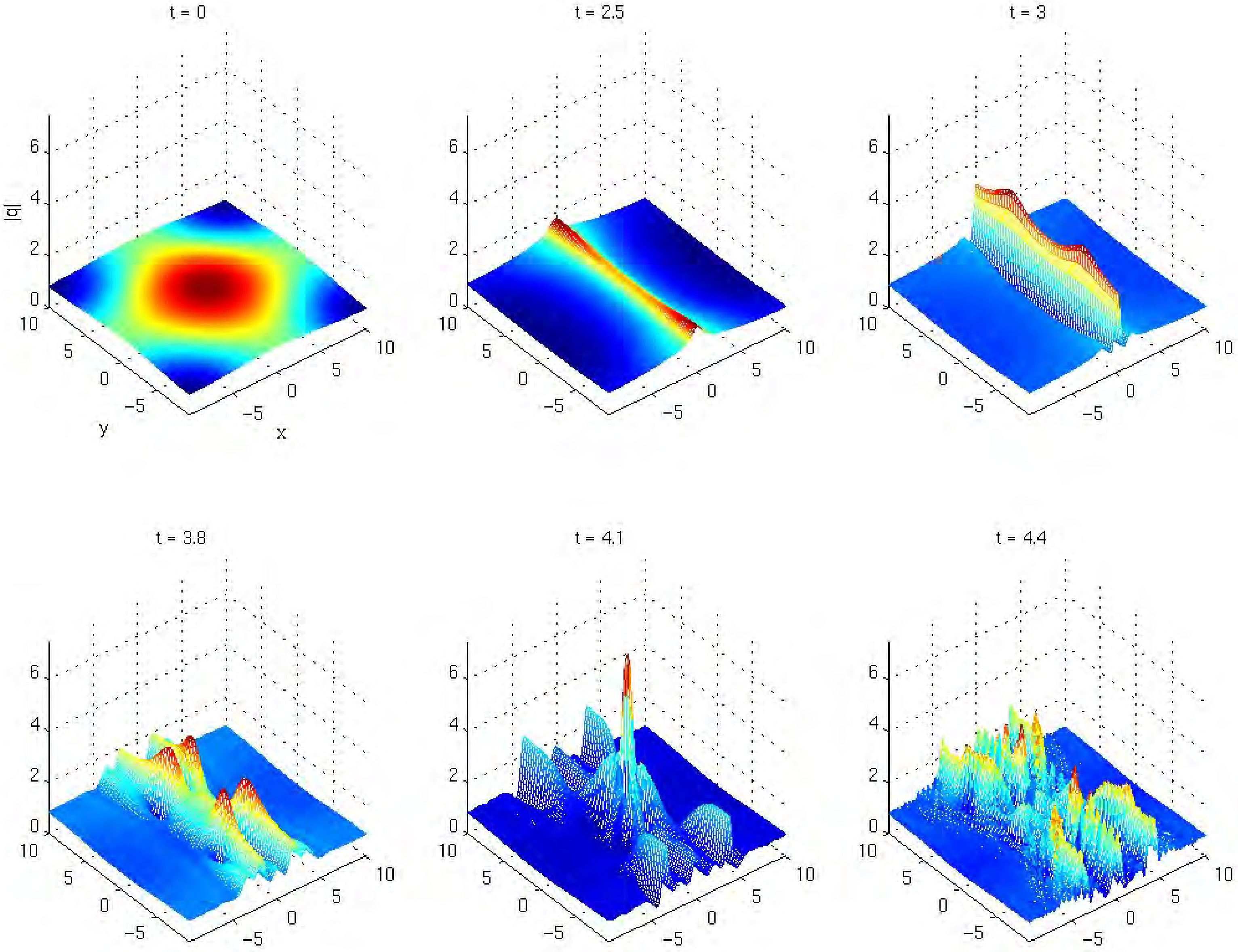


The 2+1D NLS for deep-water gravity waves possesses the property of focusing for longitudinal modulations and de-focusing for transverse modulations.

$$iq_t + q_{xx} - q_{yy} + 2|q|^2 q = 0$$

Modulational instability of oceanic waves

Development of the modulational instability



Modulational instability of oceanic waves

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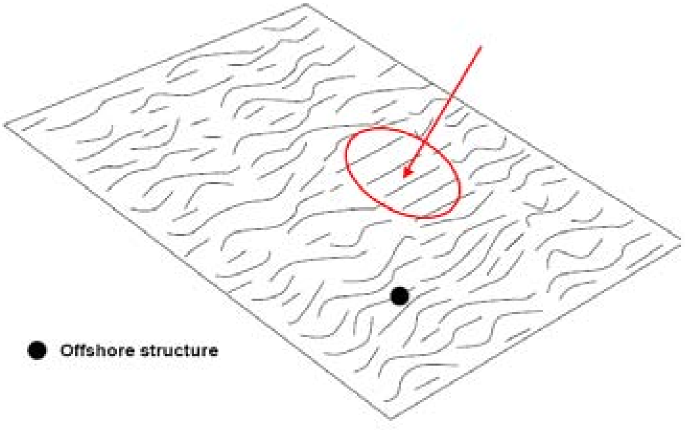

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$$iq_t + q_{xx} - q_{yy} + 2|q|^2 q = 0$$

The idea is:

Waves trapped by jet currents =>

described in modal representation

Effectively unidirectional nonlinear wave evolution

Increase of rogue wave likelihood

due to nonlinear self-modulation effects

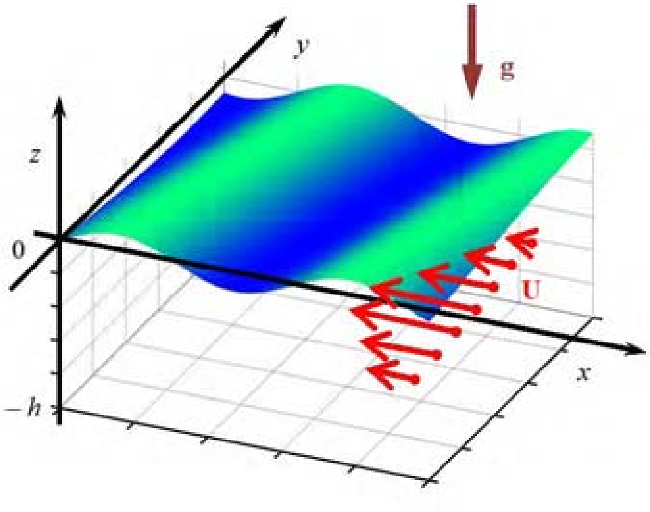
Currents: Smith'70; Peregrine&Smith'75; Basovitch'81; Edge waves: Dubinina et al,'05, '06; Pelinovsky et al,'10

Asymptotic modal theory

Basic equations

Euler equations for incompressible ideal fluid with the current $\vec{U} = (U(y$

and flow velocity components associated with the wave motion: $\vec{v} = (u + U.$



Water bulk $z \leq \eta$:

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{U} + \vec{v}, \nabla\right)\left(\vec{U} + \vec{v}\right) + \nabla P = \vec{g}$$

$$\nabla \cdot \left(\vec{U} + \vec{v}\right) = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Surface BC $z = \eta$:

$$P = 0 \quad at \quad z = \eta$$

$$\frac{\partial \eta}{\partial t} + \left(\vec{U} + \vec{v}, \nabla\right)\eta = w \quad at \quad z = \eta$$

Bottom BC :

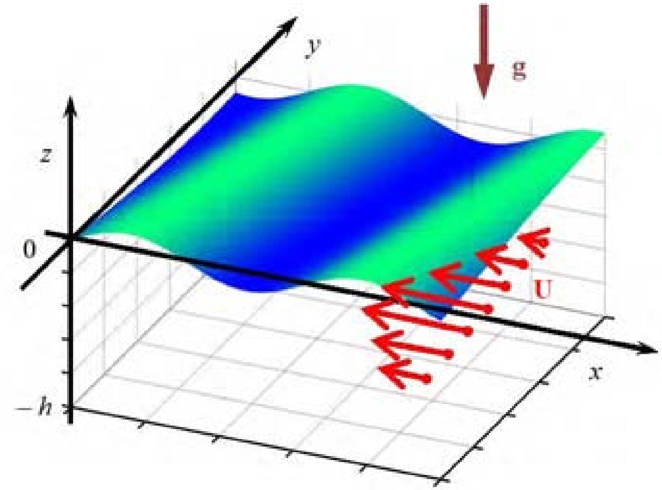
$$w \to 0 \quad when \quad z \to -\infty$$

Linear modal theory

Waves propagate predominantly along the *Ox* horizontal axis opposite to the current with the longitudinal wavenumber $k > 0$ and frequency ω.

The small parameter $\varepsilon << 1$ which has the meaning of the wave steepness will characterize weak nonlinearity of the wave motions.

Hydrodynamic fields taking into account the leading-order (linear) wave perturbations $O(\varepsilon)$:



Surface displacement:

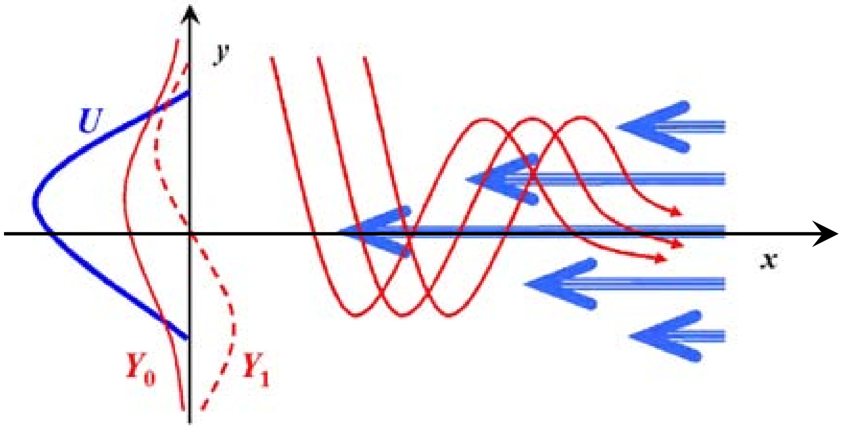
$$\eta(x,y,t) = \varepsilon A(y)\exp(i\omega t - ikx)$$

Wave velocity components:

$$u(x,y,z,t) = \varepsilon\hat{u}(y,z)\exp(i\omega t - ikx)$$
$$v(x,y,z,t) = \varepsilon\hat{v}(y,z)\exp(i\omega t - ikx)$$
$$w(x,y,z,t) = \varepsilon\hat{w}(y,z)\exp(i\omega t - ikx)$$

Excess pressure with respect to the rest condition:

$$P(x,y,z,t) = \varepsilon\hat{P}(y,z)\exp(i\omega t - ikx) - gz$$



2D BVP

The governing equations may be reduced to the 2D boundary value problem (BVP) following either of the two approaches:

1) The velocity components may be expressed in terms of the pressure from *x*-, *y*- and *z*-projections of the Euler equations. Then, the continuity equation yields the BVP on the pressure.

2) Combinations of derivatives of the Euler equation projections of the form

$$\frac{\partial}{\partial z}Euler_x - \frac{\partial}{\partial x}Euler_z \qquad \frac{\partial}{\partial z}Euler_y - \frac{\partial}{\partial y}Euler_z$$

may be used to obtain the BVP (of a slightly different form) on the vertical velocity.

The second method yields the 2D nonlinear BVP on the function $\Phi(y,z)$ and eigenfrequencies ω assuming that the longitudinal wavenumber k is given:

$$\frac{\partial^2\Phi}{\partial z^2} + \frac{\partial^2\Phi}{\partial y^2} + \left(\frac{\Omega''}{\Omega} - 2\frac{\Omega'^2}{\Omega^2} - k^2\right)\Phi = O(\varepsilon) \quad for \quad z \le 0$$

$$\frac{\partial\Phi}{\partial z} = \frac{\Omega^2}{g}\Phi + O(\varepsilon) \quad for \quad z = 0$$

$$\Phi \to 0 \quad for \quad z \to -\infty$$

The function $\Phi(y,z)$ has similarity with the velocity potential: $\hat{w}(y,z) = \frac{\partial\Phi}{\partial z}$

The function $\Omega(y) = \omega - kU(y)$ characterizes the local Doppler shift.

First appeared in [Basovitch1981]

Reduction to 1D BVP

The 2D BVP may be solved numerically, treating the condition in the water bulk as a weakly perturbed Helmholtz equation:

$$\frac{\partial^2\Phi}{\partial z^2}+\frac{\partial^2\Phi}{\partial y^2}-k^2\Phi=-\left(\frac{\Omega''}{\Omega}-2\frac{\Omega'^2}{\Omega^2}\right)\Phi \quad for \quad z\le 0$$

$$\frac{\partial\Phi}{\partial z}=\frac{\Omega^2}{g}\Phi \quad for \quad z=0$$

$$\Phi\to 0 \quad for \quad z\to-\infty$$

The 2D BVP may be reduced to 1D BVP under some extra assumptions like weak current ($\gamma << 1$) or broad current ($\mu << 1$), where $\gamma \sim \frac{\max|U|}{C_{ph}}$ $\mu \sim (kL_U)^{-1}$

An approximate separation of variables is implied:

$$\Phi(y,z)=F(y,z)Z(y,z) \qquad Z(y,z)=\exp\left(z\frac{\Omega^2}{g}\right) \qquad \left.\frac{\partial F}{\partial z}\right|_{z=0}=0$$

(the latter condition is to satisfy the surface BC)

If the current is absent, then $Z = \exp(z\omega^2/g)$, $F = \exp(ik_y y)$, and we have the linear dispersion relation for deep-water waves: $\omega^4 = g^2(k^2 + k_y^2)$.

Then, in the domain $|kz|$ ~< 1 we obtain $$\frac{\partial^2 F}{\partial y^2}+\left(\frac{\Omega^4}{g^2}-k^2\right)F=O(\gamma\mu)$$

If $\gamma\mu << 1$, the dependence of F on z may be neglected: $F(y,z) \approx Y(y)$.

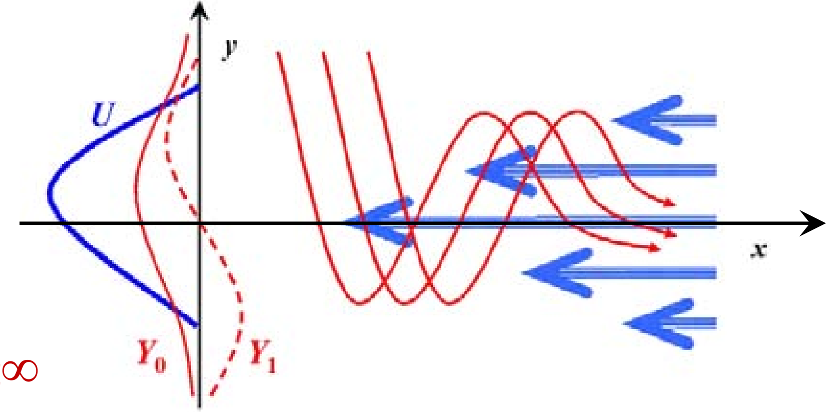
1D BVPs

Waves propagate predominantly along the *Ox* horizontal axis opposite to the current with the longitudinal wavenumber k and frequency ω. Within the linear approximation, boundary-value problem (BVP) may be formulated for the wave structure in the transverse direction *Oy* [Shrira & Slunyaev, JFM2014]:

Assumption of a broad current

$$\frac{d^2Y}{dy^2} + \frac{k^2}{\omega_g^4}\left(\Omega^4 - \omega_g^4\right)Y = 0$$

$$\Omega(y) = \omega - kU \qquad \omega_g = \sqrt{gk}$$



Decaying and non-decaying conditions at $y \to \pm\infty$ specify trapped and passing-through modes, respectively. Eigenfrequencies ω and eigenfunctions $Y(y)$ determine the modes. One eigenvalue corresponds to one eigenmode. Trapped modes correspond to the discrete spectrum which may exist only for opposite currents, $kU < 0$, and may have the values $\omega_g + \min(kU) < \omega < \omega_g$. Eigenfunctions are generally not orthogonal.

Assumption of a weak current (Sturm-Liuoville BVP)

$$\frac{d^2Y}{dy^2} + 4\frac{k^2}{\omega_g}\left(\omega - \left(\omega_g + kU\right)\right)Y = 0$$

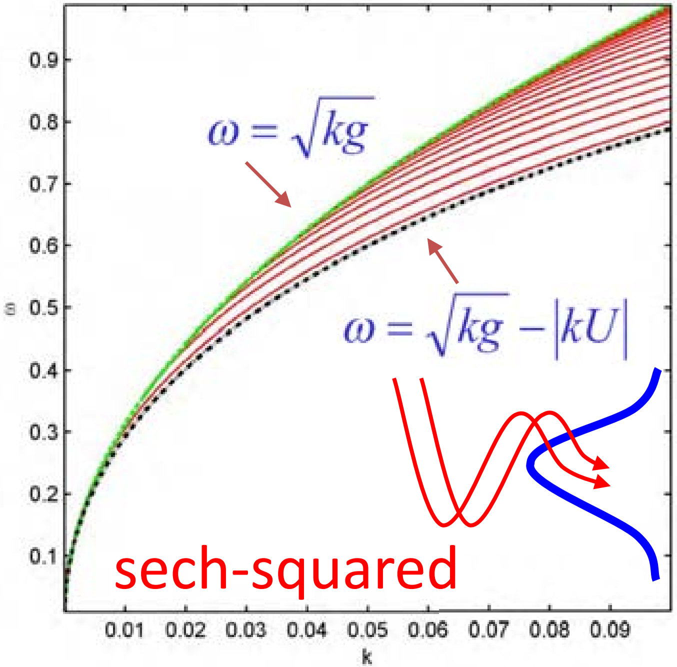
First appeared in [Peregrine & Smith, 1975]

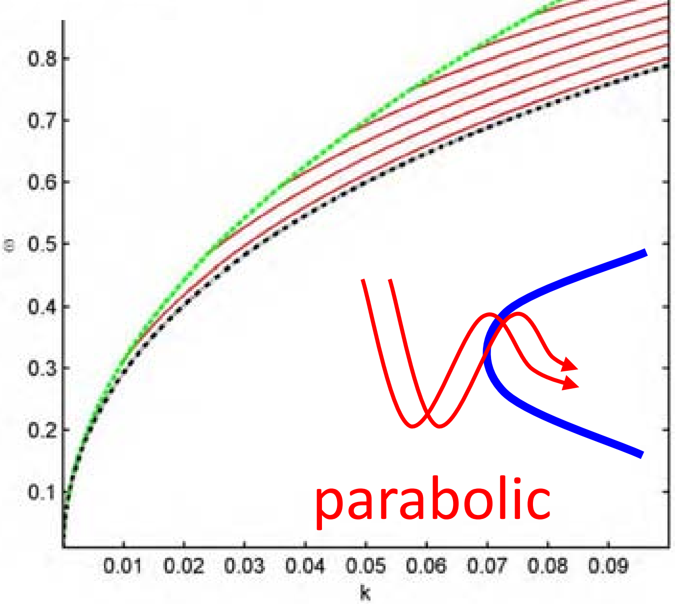
Eigenmodes of the Sturm-Liouville (SL) problem form the full orthogonal basis.
At least one trapped mode exists if $kU < 0$ for all y.

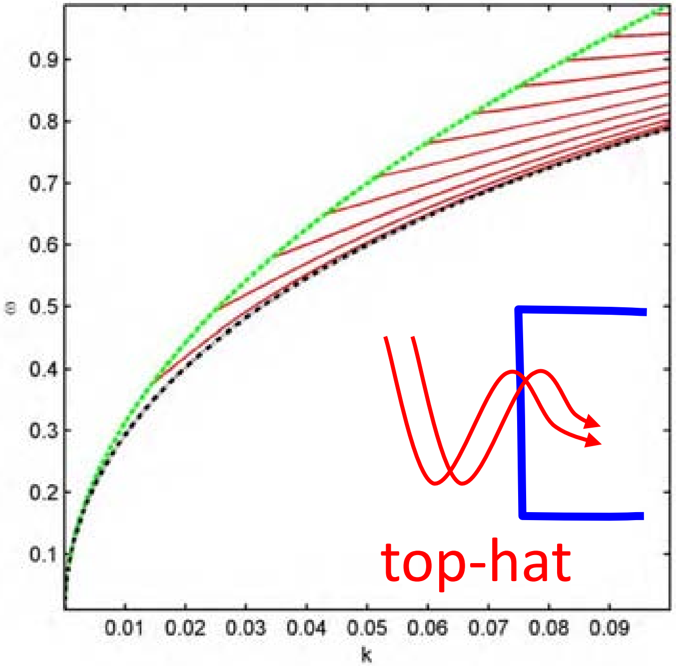
Dispersion relations for modes

For some particular shapes of the jet current the dispersion relations (solutions of the Sturm-Liouville problem) can be found analytically.

max|U| = 2 m/s,
characteristic width of the jet is 200 m







Weakly nonlinear theory:

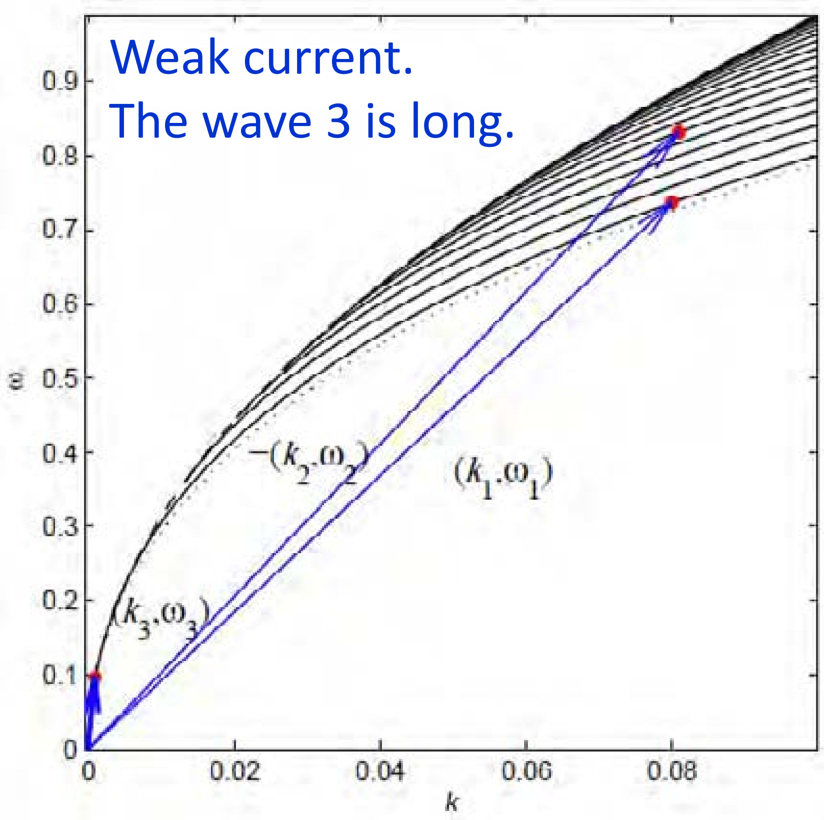
1) three-mode interactions

Allowed nonlinear wave resonances

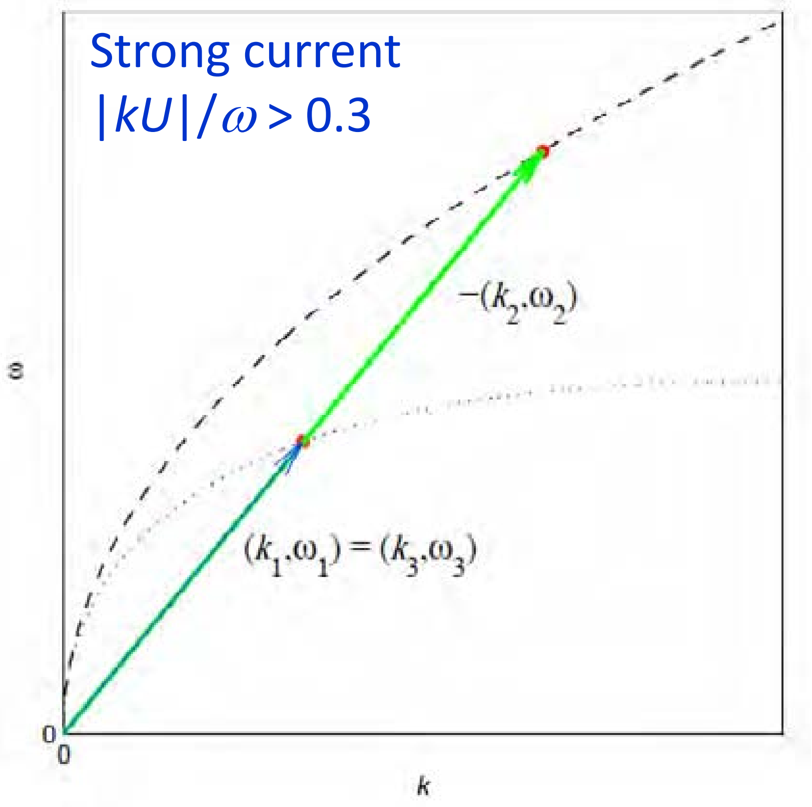
Three-wave resonant conditions become possible for waves trapped by jet currents, if one of the waves is sufficiently long or the current is sufficiently strong.

Resonant conditions for a mode triad: $k_1 + k_2 + k$ $k_2 <$

$\omega_1 + \omega_2 + \omega$ $\omega_2 <$



Weak current.
The wave 3 is long.



Strong current
$|kU|/\omega > 0.3$

Resonant conditions for quartets of modes are allowed too.

3-wave interaction theory

Assuming that the effect of nonlinearity is small, we introduce the parameter $\varepsilon << 1$ and use the asymptotic series for all hydrodynamic fields,

$$P(x,y,z,t) = -gz + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \ldots$$

$$\eta(x,y,t) = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \ldots$$

$$u(x,y,z,t) = \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \ldots$$

$$v(x,y,z,t) = \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \ldots$$

$$w(x,y,z,t) = \varepsilon w^{(1)} + \varepsilon^2 w^{(2)} + \ldots$$

To allow the mode evolution and to account the large lateral scale of the current and related fields, we use slow time and coordinates: $t \to t_0 + \varepsilon t_1$, $x \to x_0 + \varepsilon x_1$, $y \to y_0 + \varepsilon y_1$. At this stage one has to set relations between small parameters and scales. It is more convenient to derive the nonlinear theory for the variable of pressure.

After calculations, the equation in the nonlinear order $O(\varepsilon^2)$ for mode $Y_1(y_0)$ (for a broad current the dependence of modes on y is fast) has the form:

$$\frac{\partial P_1}{\partial t_1} Y_1 + V_1 \frac{\partial P_1}{\partial x_1} Y_1 - ig^2 \frac{\Omega_1'}{\Omega_1^4} P_1 Y_1' - i\left(\sigma_1^{(a)} Y_2' Y_3' + \sigma_1^{(b)} Y_2 Y_3\right) P_2^* P_3^* +$$

$$+ i \frac{g k_1}{\Omega_1 \overline{\Omega}^2} \hat{L}[k_1, \Omega_1] \boxed{Y_1^{(2)}} P_2^* P_3^* + \sum_{f:\omega_1^{(f)} \neq \omega_1} i\kappa_1^{(f)} P_1^{(f)} \boxed{Y_1^{(f)}} = 0$$

nonlinear correction to the mode structure (faster attenuation with depth)

slave mode

$$\hat{L}[k,\Omega] Y \equiv \frac{d^2 Y}{dy_0^2} + \left(\frac{\Omega^4}{g^2} - k^2\right) Y$$

operator of the BVP

$$V_j = \frac{g^2 k_j}{2\Omega_j^3} + U \qquad \overline{\Omega}^2 \equiv \Omega_1^2 + \Omega_2^2 + \Omega_3^2$$

3-wave interaction theory

The evolution equation on the amplitude $A_1(x,t)$ of three resonantly interacting modes may be written in the simplest form as

$$\frac{\partial A_1}{\partial t} + \overline{V}_1 \frac{\partial A_1}{\partial x} - i\overline{\rho}_1 A_2^* A_3^* = 0$$

$$k_1 + k_2 + k \qquad k_2 <$$

$$\omega_1 + \omega_2 + \omega \qquad \omega_2 <$$

$$\overline{V}_1 = \frac{\int_{-\infty}^{\infty} V_1 Y_1^2 dy}{\int_{-\infty}^{\infty} Y_1^2 dy} \qquad \overline{\rho}_1 = \frac{\int_{-\infty}^{\infty} \rho_1 Y_1 Y_2 Y_3 dy}{\int_{-\infty}^{\infty} Y_1^2 dy} \qquad \rho_1 = \frac{2\Omega_1\left(\Omega_2^2 + \Omega_3^2\right)\left(\Omega_1^2 - \Omega_2\Omega_3\right)}{g^2\overline{\Omega}^2} + \frac{4k_2k_3\Omega_1^2\Omega_2\Omega_3 - \left(k_2\Omega_3^2 + k_3\Omega_2^2\right)^2}{2\Omega_1\Omega_2\Omega_3\overline{\Omega}^2}$$

$$\overline{\Omega}^2 \equiv \Omega_1^2 + \Omega_2^2 + \Omega_3^2$$

The equations on modes A_2 and A_3 are obtained by permutation of the indices (1,2,3). The derivation of this theory is not potential, but the resulting equation coefficients do not depend on vorticity. The nonlinear interaction coefficient is valid for strong currents. The 3-wave system is completely integrable.

In the limit of a weak current $\gamma << 1$ (one of the wave is very long) the equation on the short wave reads:

$$\frac{\partial A_1}{\partial t} + \overline{V}_1 \frac{\partial A_1}{\partial x} - i\overline{\rho}_1 A_{LW}^* A_2^* = 0 \qquad I_1 = \frac{\int_{-\infty}^{\infty} Y_{LW} Y_1 Y_2 dy}{\int_{-\infty}^{\infty} Y_1^2 dy}$$

$$\overline{V}_1 = \frac{g^2 k_1}{\omega_1^3} + \boxed{\gamma \frac{5}{2} U} + O\left(\gamma^2\right) = \frac{\sqrt{gk}}{2k} + U + O\left(\gamma^2\right) \qquad \overline{\rho}_1 = \frac{\omega_1^3}{g^2} I_1 + O(\gamma)$$

classic correction term of the wave velocity

3-w nonlinear effects weaken when the current becomes weaker

Weakly nonlinear theory:

4 – four-mode interactions

4-wave interaction theory

The picture of possible nonlinear interactions between quartets of modes is rich. Under the assumption of a weak current, the potential theory for water motions may be used, what greatly simplifies the derivation.

$$k_1 + k_2 = k$$

$$\omega_1 + \omega_2 = \omega$$

$$i\omega_1 \frac{\partial A_1}{\;} = \overline{\alpha}_1 A_1 |A_1|^2 + \sum \overline{\alpha}_{1j} A_1 |A$$

$$i\omega_2 \frac{\partial A_2}{\;} = \overline{\alpha}_2 A_2 |A_2|^2 + \sum \overline{\alpha}_{2j} A_2 |$$

$$i\omega_3 \frac{\partial A_3}{\;} = \overline{\alpha}_3 A_3 |A_3|^2 + \sum \overline{\alpha}_{3j} A_3 |$$

$$i\omega_4 \frac{\partial A_4}{\;} = \overline{\alpha}_4 A_4 |A_4|^2 + \sum \overline{\alpha}_{4j} A_4 |$$

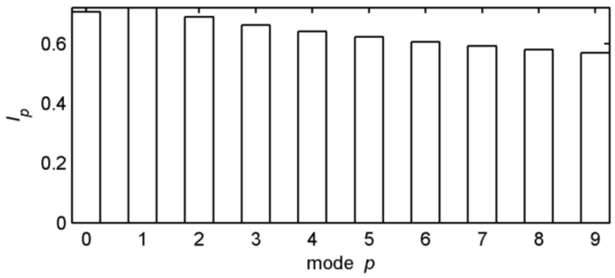
$$\overline{\alpha}_j = \alpha_j I_j \qquad \alpha_j = \frac{1}{2} k_j^2 \qquad I_j = \frac{\int_{-\infty}^{\infty} Y_j^4 dy}{\int_{-\infty}^{\infty} Y_j^2 dy}$$

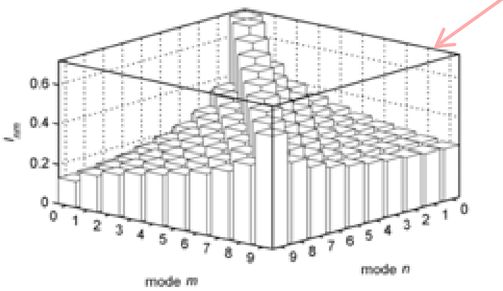
$$\overline{\alpha}_{jq} = 2\alpha_j I_{jq} \begin{cases} \sqrt{\frac{k_j}{k_q}} & если \quad k_q > k_j \\ \sqrt{\frac{k_q}{k_j}} & если \quad k_j > k_q \end{cases} \qquad I_{jq} = \frac{\int_{-\infty}^{\infty} Y_j^2 Y_q^2 dy}{\int_{-\infty}^{\infty} Y_j^2 dy}$$

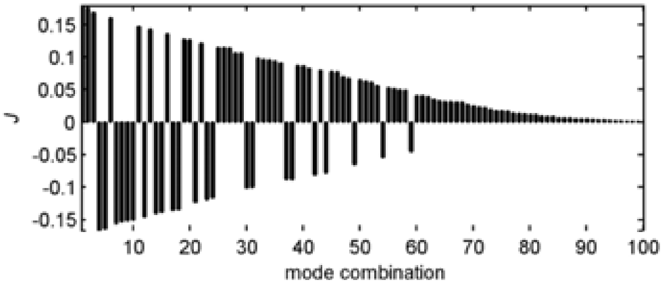
agrees with Lavrova (1983)

$$\overline{\alpha}_{jq} < 2\overline{\alpha}_j$$

$$\overline{\nu}_j = \nu_j (k_1, k_2, k_3, k_4) J_j \qquad J_j = \frac{\int_{-\infty}^{\infty} Y_1 Y_2 Y_3 Y_4 dy}{\int_{-\infty}^{\infty} Y_j^2 dy}$$







4-wave interaction theory

The picture of possible nonlinear interactions between quartets of modes is rich. Under the assumption of a weak current, the potential theory for water motions may be used, what greatly simplifies the derivation.

$$k_1 + k_2 = k$$

$$\omega_1 + \omega_2 = \omega$$

$$i\omega_1 \frac{\partial A_1}{\partial t} = \bar{\alpha}_1 A_1 |A_1|^2 + \sum \bar{\alpha}_{1j} A_1 |$$

$$i\omega_2 \frac{\partial A_2}{\partial t} = \bar{\alpha}_2 A_2 |A_2|^2 + \sum \bar{\alpha}_{2j} A_2 |$$

$$i\omega_3 \frac{\partial A_3}{\partial t} = \bar{\alpha}_3 A_3 |A_3|^2 + \sum \bar{\alpha}_{3j} A_3 |$$

$$i\omega_4 \frac{\partial A_4}{\partial t} = \bar{\alpha}_4 A_4 |A_4|^2 + \sum \bar{\alpha}_{4j} A_4 |$$

$$\bar{\alpha}_j = \alpha_j I_j \qquad \alpha_j = \frac{1}{2} k_j^2 \qquad I_j = \frac{\int_{-\infty}^{\infty} Y_j^4 dy}{\int_{-\infty}^{\infty} Y_j^2 dy}$$

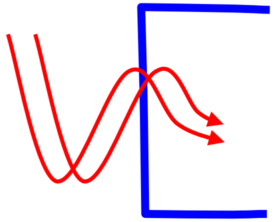
$$\bar{\alpha}_{jq} = 2\alpha_j I_{jq} \begin{cases} \sqrt{\dfrac{k_j}{k_q}} & \text{если} \quad k_q > k_j \\ \sqrt{\dfrac{k_q}{k_j}} & \text{если} \quad k_j > k_q \end{cases} \qquad I_{jq} = \frac{\int_{-\infty}^{\infty} Y_j^2 Y_q^2 dy}{\int_{-\infty}^{\infty} Y_j^2 dy}$$

agrees with Lavrova (1983)

$$\bar{\alpha}_{jq} < 2\bar{\alpha}_j$$

$$\bar{\nu}_j = \nu_j(k_1, k_2, k_3, k_4) J_j \qquad J_j = \frac{\int_{-\infty}^{\infty} Y_1 Y_2 Y_3 Y_4 dy}{\int_{-\infty}^{\infty} Y_j^2 dy}$$

Example of a top-hat current



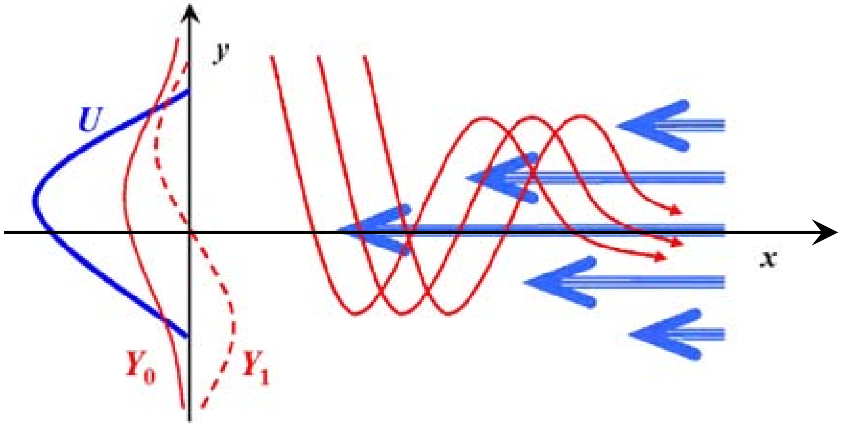
$$I_j = \begin{cases} 1, & j = 0 \\ \dfrac{3}{4}, & j \neq 0 \end{cases} \qquad I_{jq} = \begin{cases} I_j, & j = q \\ \dfrac{1}{2}, & j \neq q \end{cases}$$

$$J_j = 0 \Rightarrow \frac{\partial}{\partial t} |A_j|^2 = 0$$

NLSE for one trapped mode

The nonlinear Schrodinger equation (NLSE) for a single mode accounting for 4-wave nonlinear interaction within the non-potential theory for a broad current reads

$$i\left(\frac{\partial\psi}{\partial t}+\overline{V}\frac{\partial\psi}{\partial x}\right)+\overline{\beta}\frac{\partial^2\psi}{\partial x^2}+\overline{\alpha}|\psi|^2\psi=0$$



$$\overline{V}=\frac{\int_{-\infty}^{\infty} VY^2dy}{\int_{-\infty}^{\infty} yY^2d} \qquad \overline{\alpha}=\frac{\int_{-\infty}^{\infty} \alpha Y^4d}{\int_{-\infty}^{\infty} yY^2d} \qquad \overline{\beta}=\frac{\int_{-\infty}^{\infty} \beta Y^2d}{\int_{-\infty}^{\infty} yY^2d}$$

$$V=U+\frac{kg^2}{2\Omega^3} \qquad \alpha=\frac{\Omega\left(-h^4+5k^4+2h^2k^2\right)}{12h^2} \qquad \beta=\frac{\Omega\left(3k^2-2h^2\right)}{8h^4} \qquad \Omega^2=gh$$

The equation describes a mode with frequency ω and corresponding mode $Y(y)$. The coefficients are valid for a broad but may be strong current.

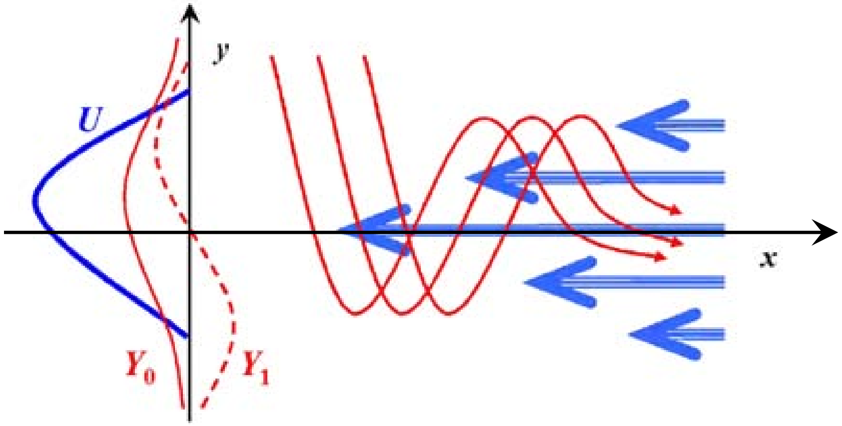
The coefficients may be further simplified assuming that the $V(y)$, $\alpha(y)$ and $\beta(y)$ are slower functions of the transverse coordinate than the mode $Y(y)$.

In the limit of a weak current the coefficients are greatly simplified too, and the potential theory may be applied.

NLSE for one trapped mode

The nonlinear Schrodinger equation (NLSE) for a single mode accounting for 4-wave nonlinear interaction was presented in [Shrira & Slunyaev, PRE2014] (derived within the potential framework):

$$i\left(\frac{\partial\psi}{\partial t}+\overline{V}\frac{\partial\psi}{\partial x}\right)+\beta\frac{\partial^2\psi}{\partial x^2}+\overline{\alpha}|\psi|^2\psi=0$$



$$\overline{V}=\frac{1}{2}\frac{kg^2}{\omega^3}+\frac{5}{2}\overline{U} \qquad \overline{U}=\frac{\int\limits_{-\infty}^{\infty}UY^2dy}{\int\limits_{-\infty}^{\infty}Y^2dy}$$

$$\beta=\frac{\omega}{8k^2} \qquad \overline{\alpha}=\alpha I \qquad \alpha=\frac{\sqrt{gk}k^2}{2}I \qquad I=\frac{\int_{-\infty}^{\infty}Y^4dy}{\int_{-\infty}^{\infty}Y^2dy}$$

The coefficients are valid for a weak current.

The corresponding surface displacement reads:

$$\eta(x,y,t)=\frac{1}{2}\psi(x,t)Y(y)\exp(i\omega t-ikx)+c.c.$$

The general solution may be represented in the form of mode superposition relying on the orthogonality property of the Sturm-Liouville problem eigenfunctions.

Numerical scheme

Equations of hydrodynamics

We simulate homogeneous inviscid water of the constant density under the action of gravity force. The basin is infinitely deep.

The jet current flows along the *Ox* horizontal coordinate and depends on the transverse coordinate *y* only: $\mathbf{U}=(U(y),0,0)$.

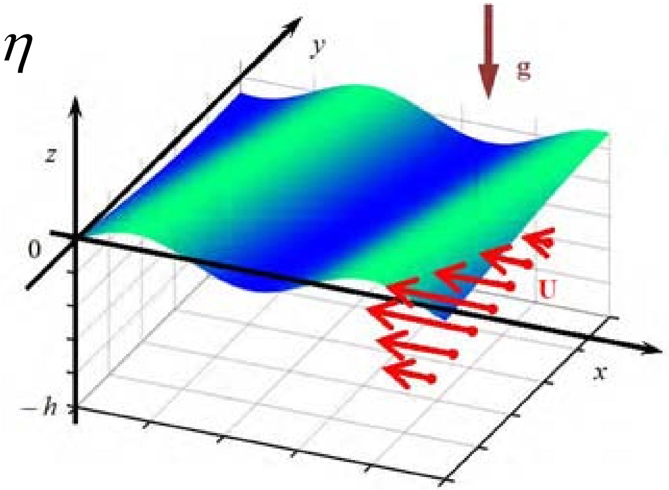
The fluid motions are assumed to be potential, so that the velocity $\mathbf{v} = (u,v,w) = \mathbf{U} + \nabla\varphi(x,y,z,t)$. At the free surface $z = \eta(x,y,t)$ the surface velocity potential is defined by $\Phi(x,y,t) = \varphi(x,y,z=\eta,t)$.

$$\eta_t + (\nabla\Phi + \mathbf{U})\cdot\nabla\eta - w\left(1+(\nabla\eta)^2\right) = 0, \quad z=\eta$$

$$\Phi_t + g\eta + \frac{1}{2}(\nabla\Phi+\mathbf{U})^2 - \frac{1}{2}w^2\left(1+(\nabla\eta)^2\right) + P = 0, \quad z=\eta$$

$$\nabla\varphi = 0, \quad z \le \eta$$

$$\varphi \to 0, \quad z \to -\infty$$



$$\Phi = \varphi(x,y,z=\eta,t)\Big| \qquad w(x,y,t) = \frac{\partial\varphi(x,z,t)}{\partial z}\Big|_{z=\eta(x,t)}$$

Rest conditions

When waves are absent, $\varphi = 0$, the stationary condition with zero elevation, $\eta = 0$, is supported by inhomogeneous distribution of the pressure on surface:

$$P = \overline{P} + P_a \qquad \overline{P} = -\frac{1}{2}|\mathbf{U}|^2$$

The atmosphere pressure is assumed to be zero, $P_a = 0$.

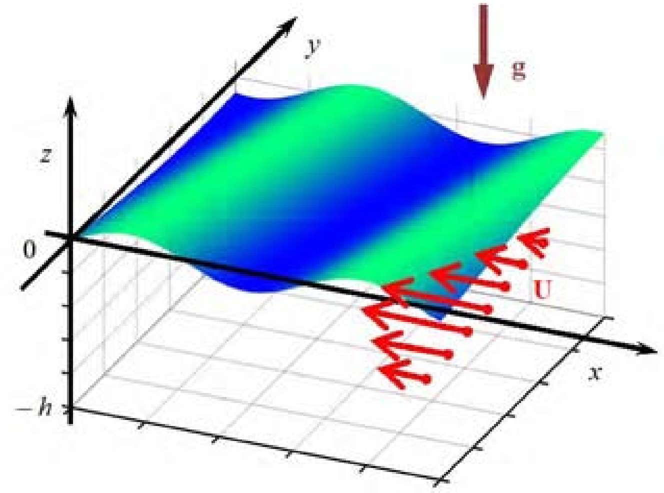
Integrals of motion

Mass $$M = \iint \eta dxdy = Const$$

Flux $$F = \iint \eta_t dxdy = Const$$

Momentum $$\mathbf{P} = \iint \eta(\mathbf{U} + \nabla\Phi) dxdy$$

Energy $$E = W^k + W^p + \int_0^t A dt = Const$$



$$W^k = \frac{1}{2}\iint \left[\Phi\eta_t + \eta|\mathbf{U}|^2 - \Phi\mathbf{U}\cdot\nabla\eta\right] dxdy \qquad W^p = \frac{1}{2}\iint \left[g\eta^2\right] dxdy \qquad A = \iint \overline{P}\eta_t dxdy$$

Numerical approach

The High Order Spectral Method [West et al., 1987] has been modified to simulate the problem. Within this approach the problem with variable surface is replaced to the one with the upper boundary defined by the rest water level $\eta = 0$. The velocity potential is represented in the form of a linear superposition of the analytic solutions of the Laplace equation with decaying condition at great depths and specified values at the water rest level $\varphi(x,y,z=0,t)$. The surface velocity potential $\Phi(x,y,t)$ and the vertical velocity on the water surface $w(x,y,z=\eta,t)$ are related to the values $\varphi(x,y,z=0,t)$ using the Taylor expansions near $z=0$ of the order M.

Velocity potential

$$\varphi = \sum_{m=1}^{M} \varphi^{(m)} \qquad \varphi^{(1)} = \Phi \qquad \varphi^{(m)} = -\sum_{k=1}^{m-1} \frac{\eta^k}{k!} \left. \frac{\partial^k \varphi^{(m-k)}}{\partial z^k} \right|_{z=0}$$

Vertical velocity

$$w = \sum_{m=1}^{M} w^{(m)} \qquad w^{(m)} = \sum_{k=0}^{m-1} \frac{\eta^k}{k!} \left. \frac{\partial^{k+1} \varphi^{(m-k)}}{\partial z^{k+1}} \right|_{z=0}$$

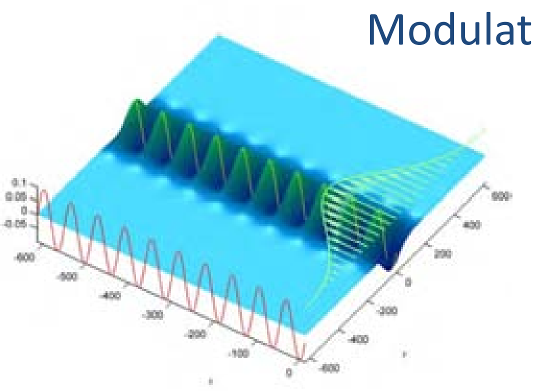
The boundary conditions are periodic along *Ox* and *Oy*.

The simulations were performed for $M = 3$ and $M = 4$ (up to 4- and 5-wave interactions were resolved).

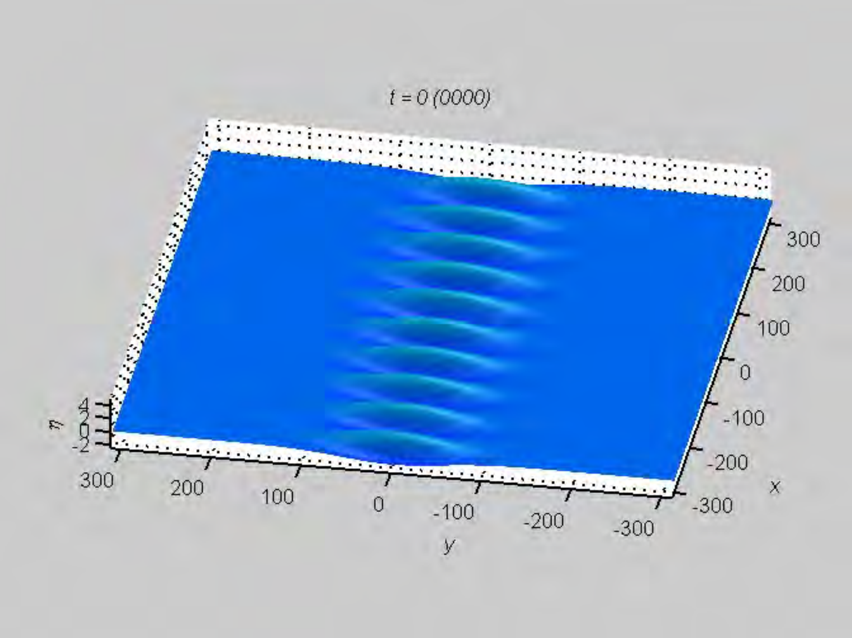
The maximum deviation of the total energy is below 0.05%.

Results of numerical simulations

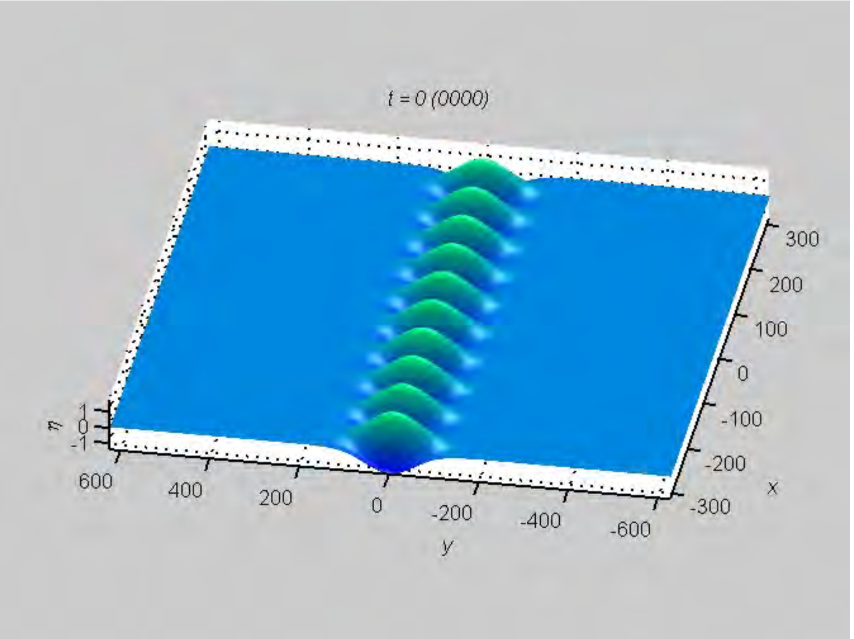
Examples of regular and modulated waves



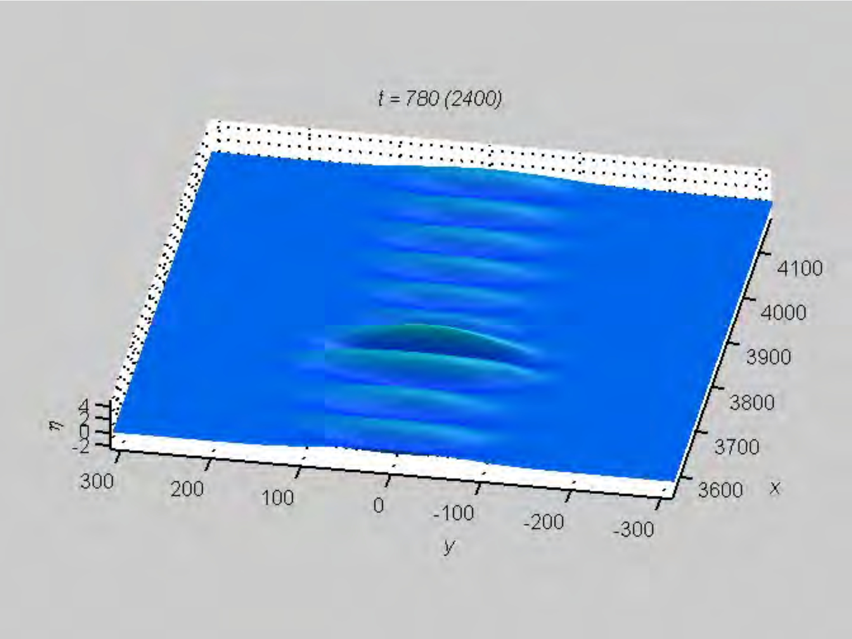
Modulational instability of a weakly modulated trapped wave (fundamental mode)



DNS of a uniform trapped wave (fundamental mode)



Terminal stage of the modulational instability leading to wave overturning



Ini. condition – Envelope soliton of trapped waves

The nonlinear Schrodinger equation possesses exact solutions in the form of envelope solitons:

$$\psi_{es}(x,t)=A\frac{\exp\left(i\frac{k^2A^2}{4}I^2\omega t\right)}{\cosh\left(\sqrt{2}k^2AI(x-Vt)\right)}=\frac{a}{I}\frac{\exp\left(i\frac{k^2a^2}{4}\omega t\right)}{\cosh\left(\sqrt{2}k^2a(x-Vt)\right)}$$

$$I^2=\frac{\int_{-\infty}^{\infty}Y^4dy}{\int_{-\infty}^{\infty}Y^2dy}\le 1$$

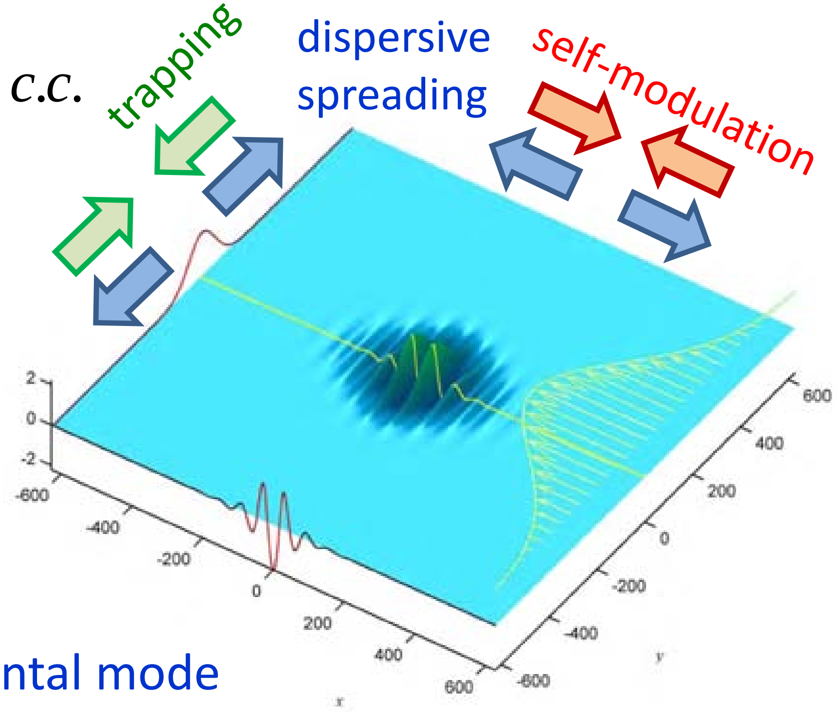
$$\eta_{es}(x,y,t)=\frac{1}{2}\psi(x,t)Y(y)\exp(i\omega t-ikx)+c.c.$$

$$\Phi_{es}(x,y,t)=\frac{ig}{2\omega_g}\psi(x,t)Y(y)\exp(i\omega t-ikx)+c.c.$$

The idea is that:

i) the self-modulation effect is not relaxed due to the transverse wave de-modulation; coherent structures like solitons live longer;

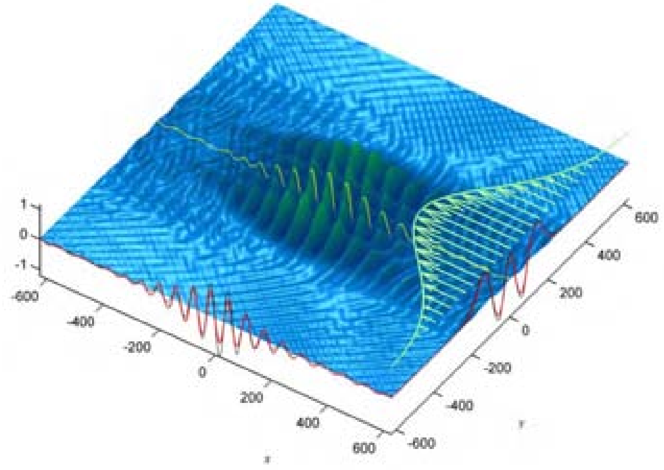
ii) long-lived coherent structures result in greater probability of high waves.



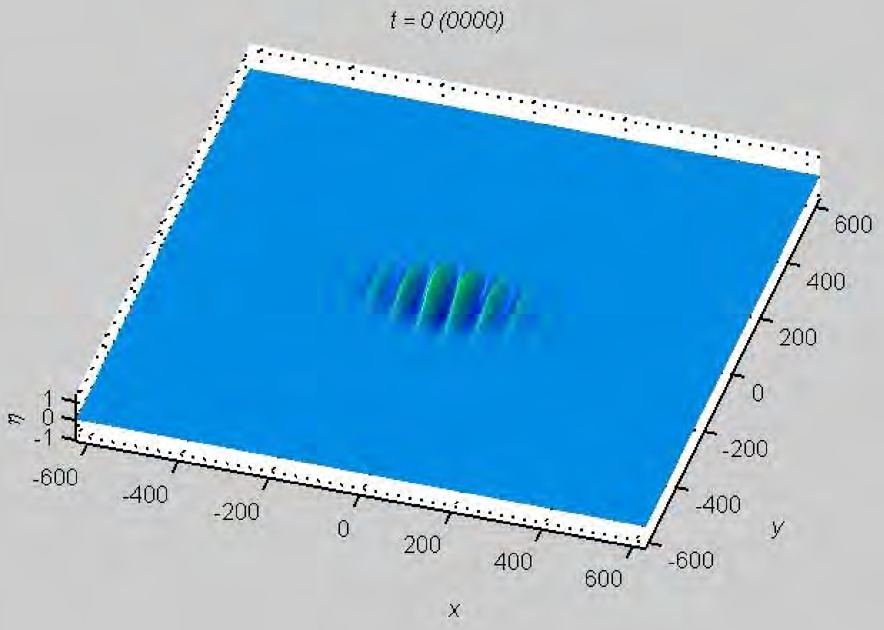
Example of a soliton of fundamental mode with the steepness *kA* = 0.2

Envelope solitons on jet currents

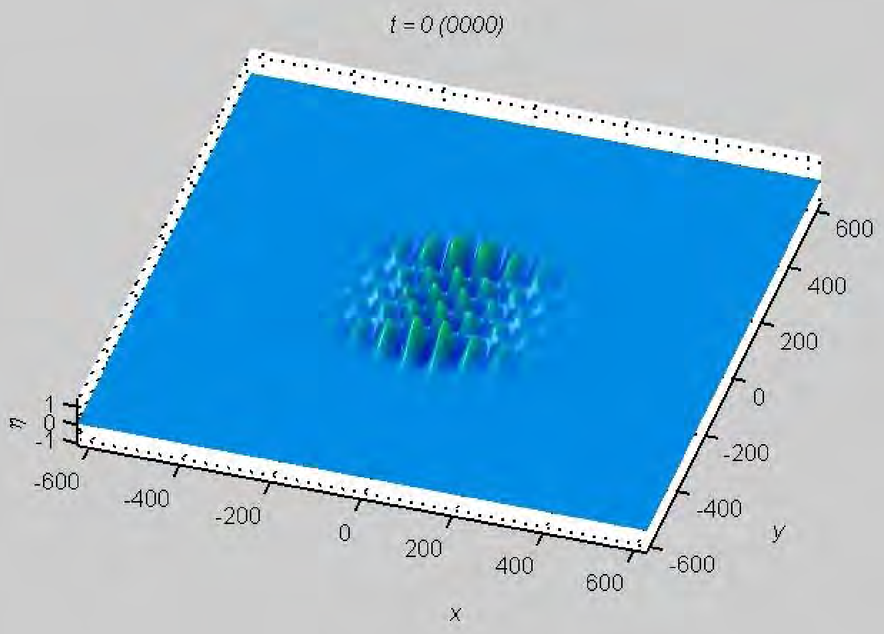
The initial conditions according to the weakly nonlinear analytic theory are simulated within the Euler equations (HOSM)



Fundamental mode (n = 1)



Firth mode (n = 5)



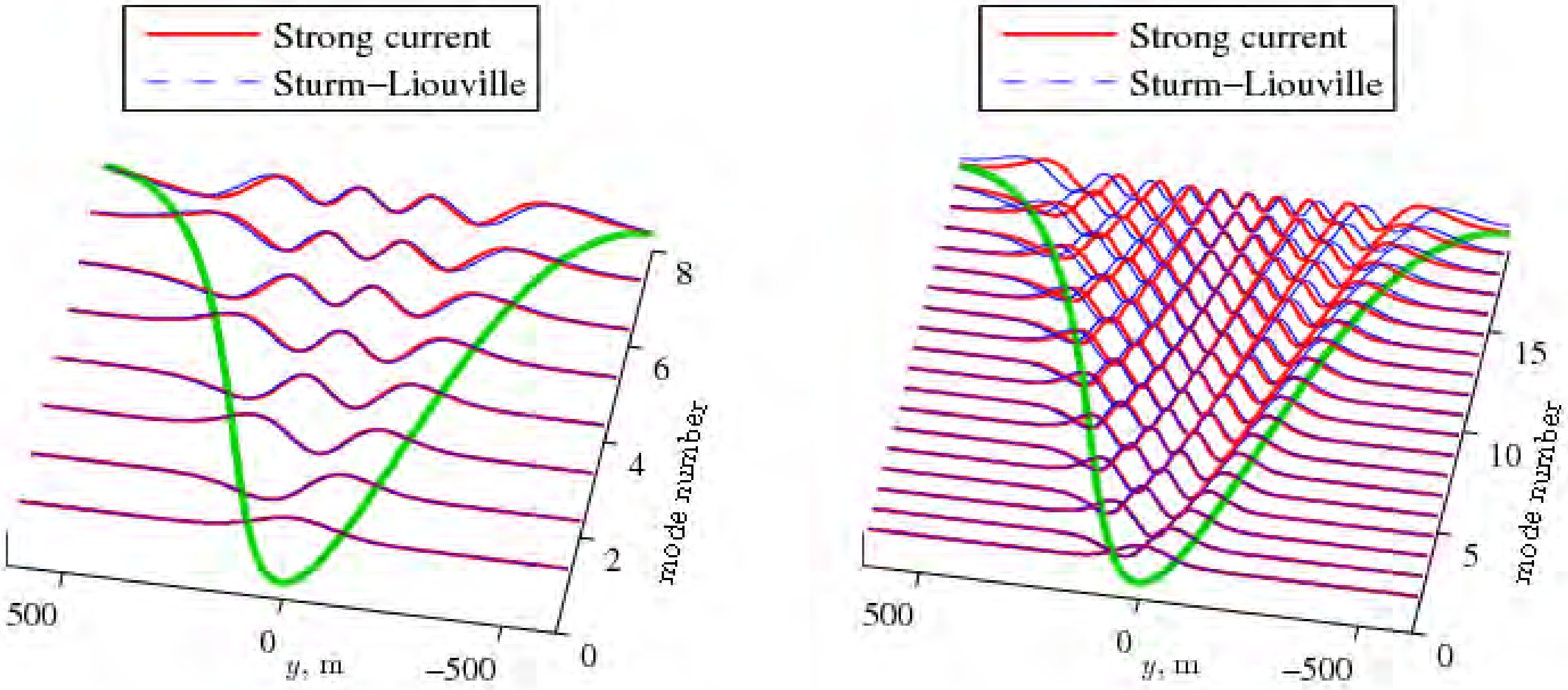
The jet current in the numerical simulations

A periodic current shape is used to fit the periodic boundary condition:

$$U(y) = U_0 cn^2\left(2K\frac{y}{L_y}, s^2\right)$$

where $K(s^2)$ is the complete elliptic integral of the first kind with the parameter $s = 0.9$; the current speed is $U_0 = -2$ m/s.

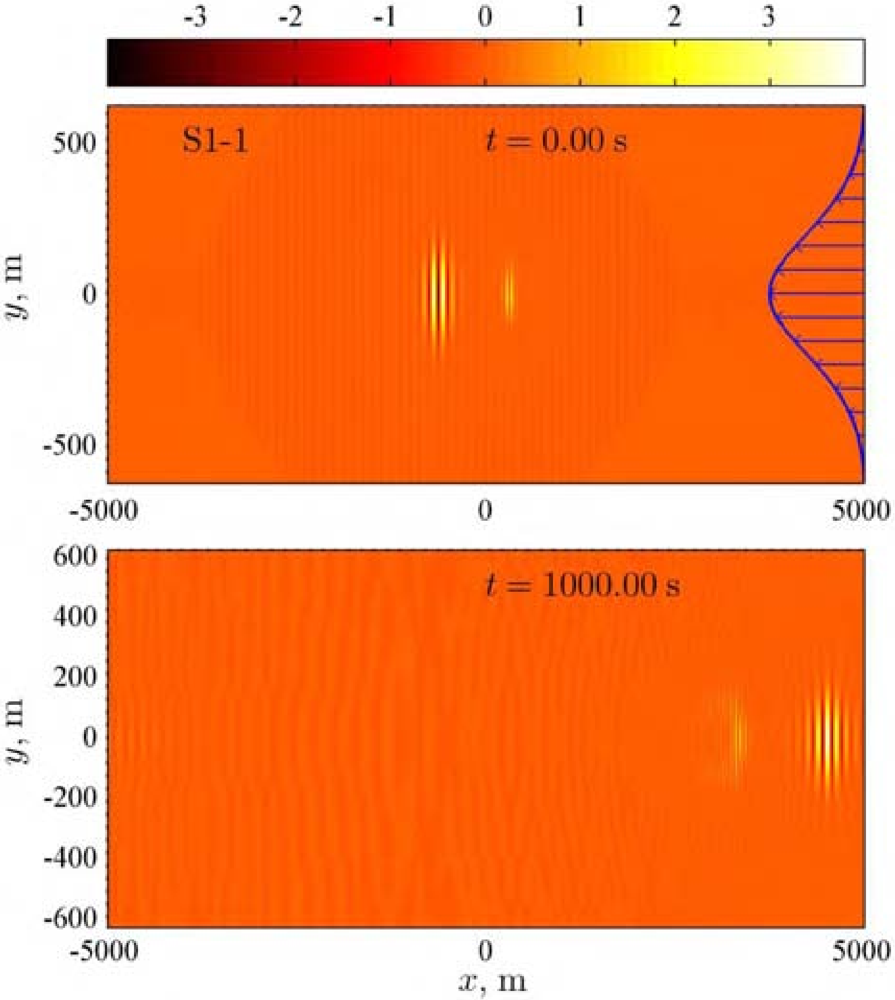
The current and modes for $k = 0.05$ rad/m (left) and $k = 0.1$ rad/m (right):



Interaction between solitons of fundamental mode

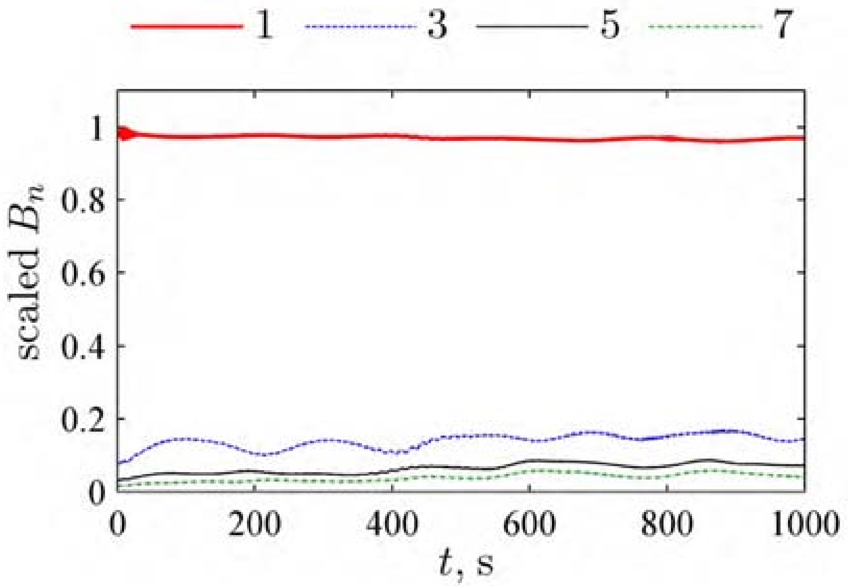
k_1 = 0.05 rad/m, k_2 = 0.1 rad/m, $k_1 A_1 = k_2 A_2$ = 0.2.

Wave periods according to the solution of the SL problem: T_1 = 10.3 s, T_2 = 7.9 s.



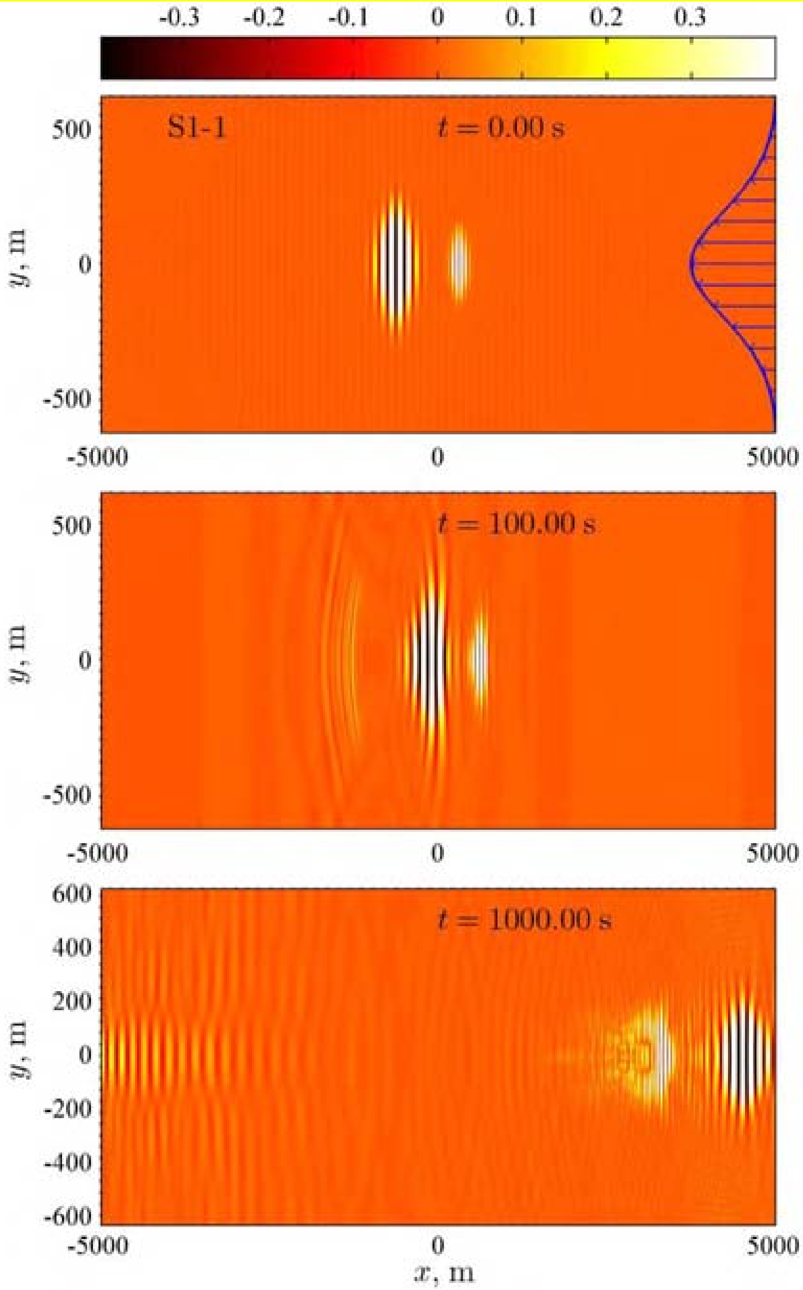
Integral mode amplitudes:

$$B_n(t)=\sqrt{\int b_n^2 dx} \qquad b_n(x,t)=\frac{\int \eta Y_n dy}{\int Y_n^2 dy}$$



Sturm-Liouville BVP for k = 0.05 rad/m

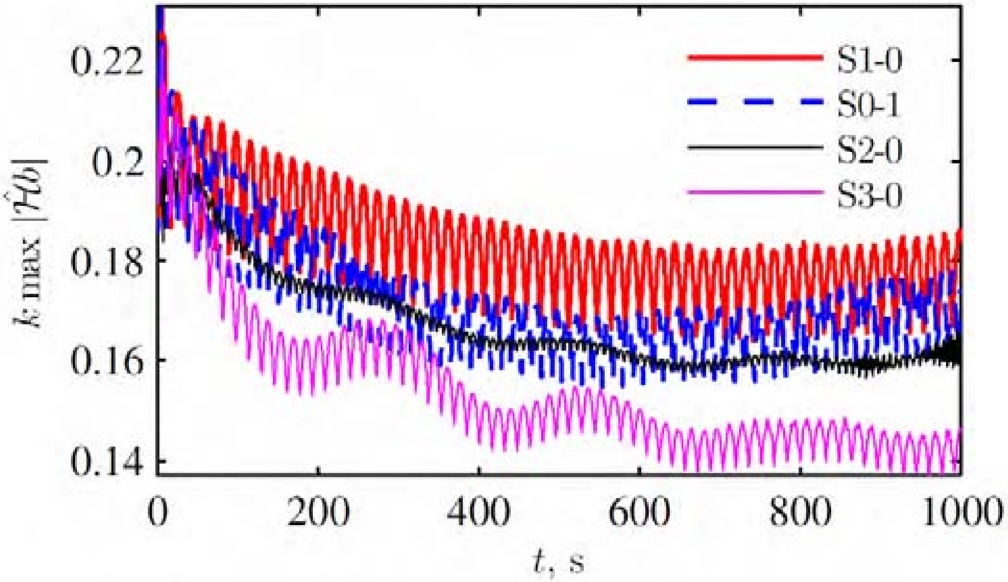
Interaction between solitons of fundamental mode



Small-amplitude waves are generated by the inaccurate initial condition and in the course of the soliton collision.

In these surface snapshots the displacements are 10 times magnified.

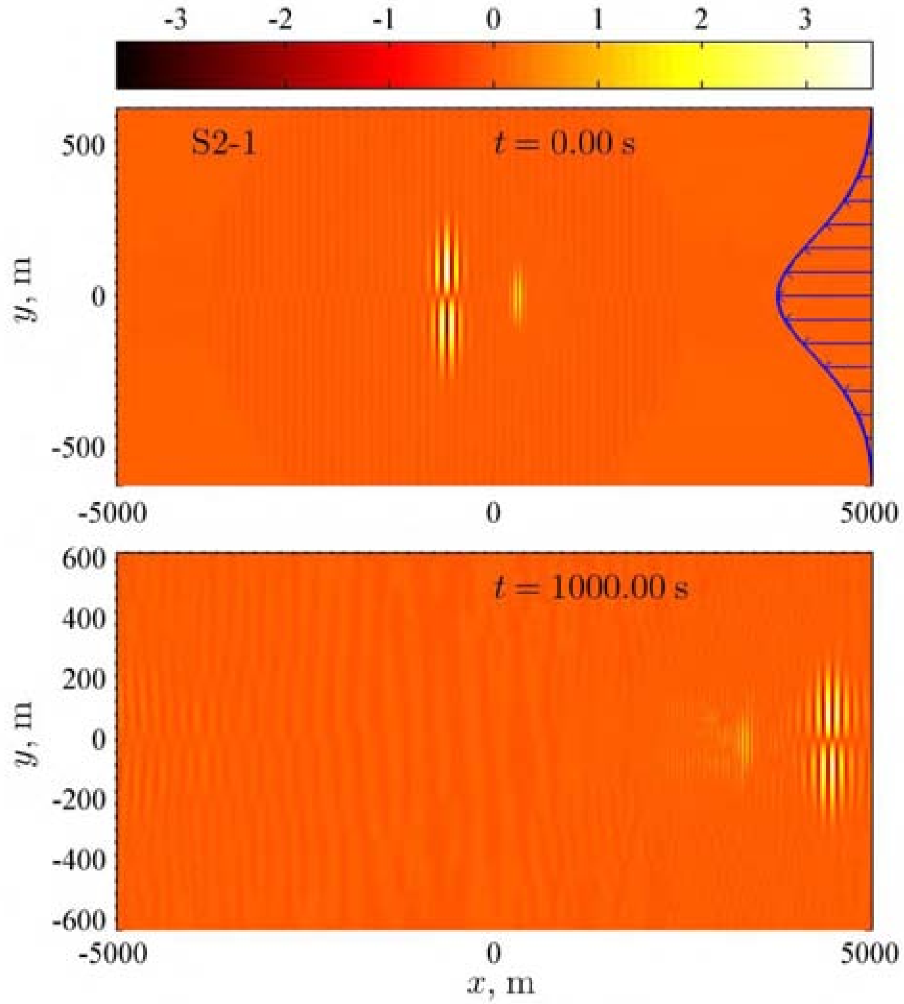
Generated solitary patters have somewhat smaller steepness than initially, in the range *kA* = 0.14…0.18.



Interaction between solitons of different modes

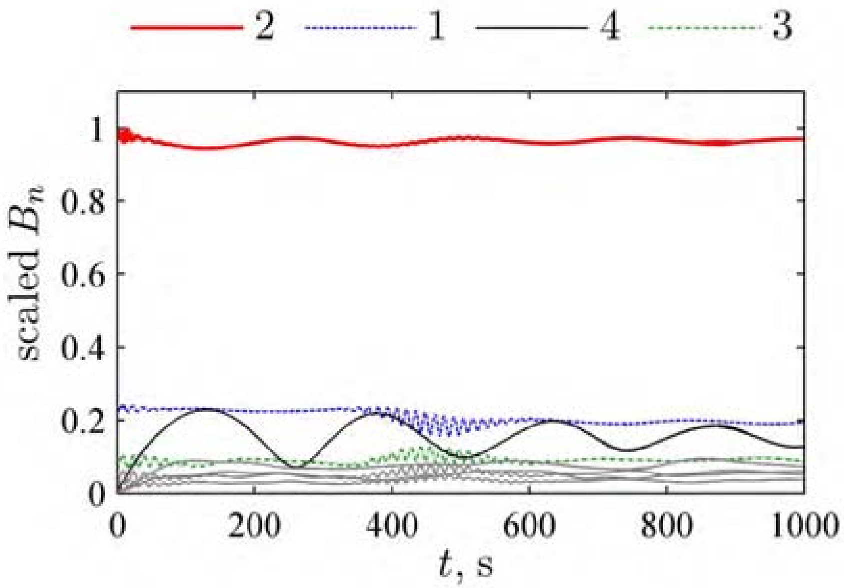
$k_1 = 0.05$ rad/m & $n_1 = 2$, $k_2 = 0.1$ rad/m & $n_2 = 1$, $k_1A_1 = k_2A_2 = 0.2$.

Wave periods according to the solution of the SL problem: $T_1 = 10.0$ s, $T_2 = 7.9$ s.



Second mode and fundamental mode

Integral mode amplitudes
for $k = 0.05$ rad/m

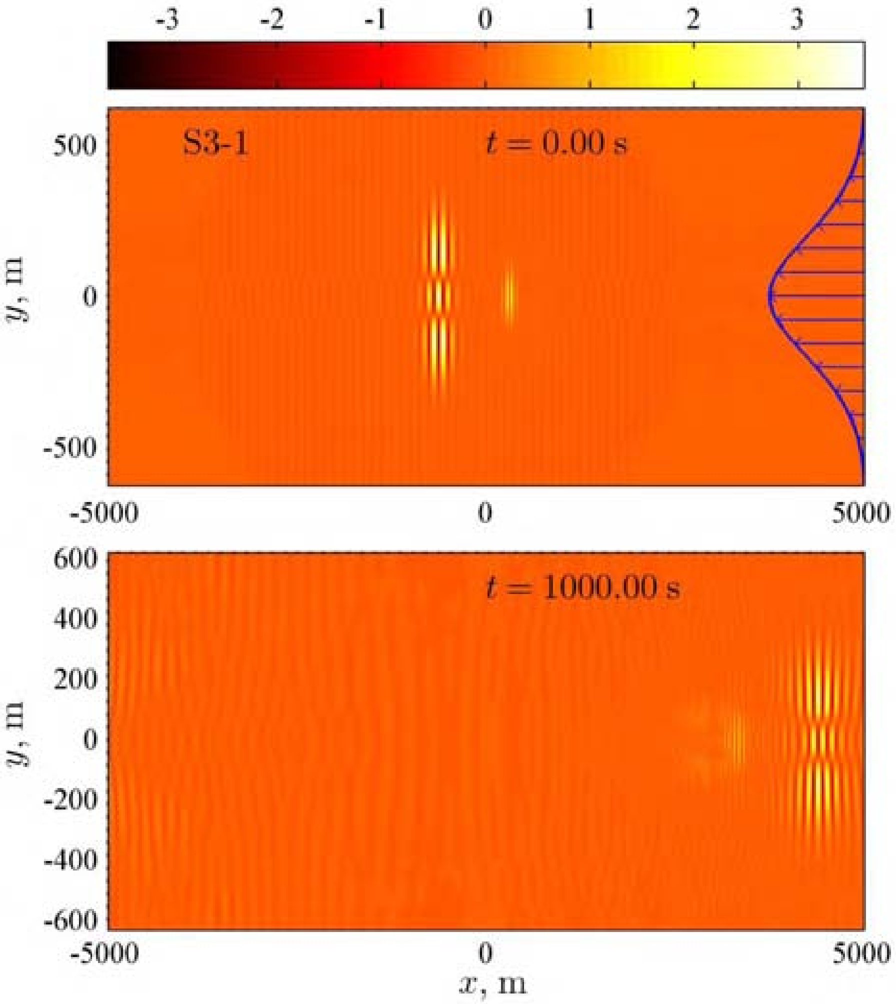


$$B_n(t) = \sqrt{\int b_n^2 dx} \qquad b_n(x,t) = \frac{\int \eta Y_n dy}{\int Y_n^2 dy}$$

Interaction between solitons of different modes

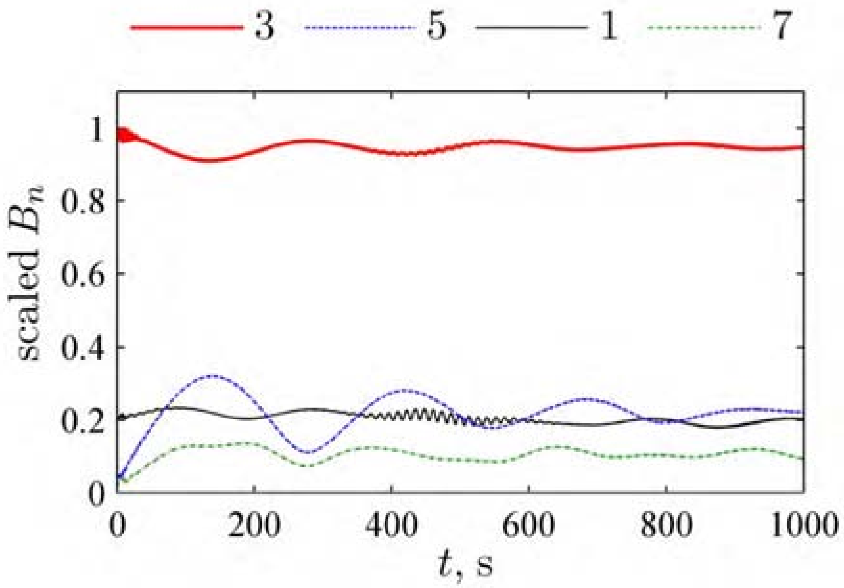
k_1 = 0.05 rad/m & n_1 = 3, k_2 = 0.1 rad/m & n_2 = 1, $k_1A_1 = k_2A_2$ = 0.2.

Wave periods according to the solution of the SL problem: T_1 = 9.8 s, T_2 = 7.9 s.



Third mode and fundamental mode

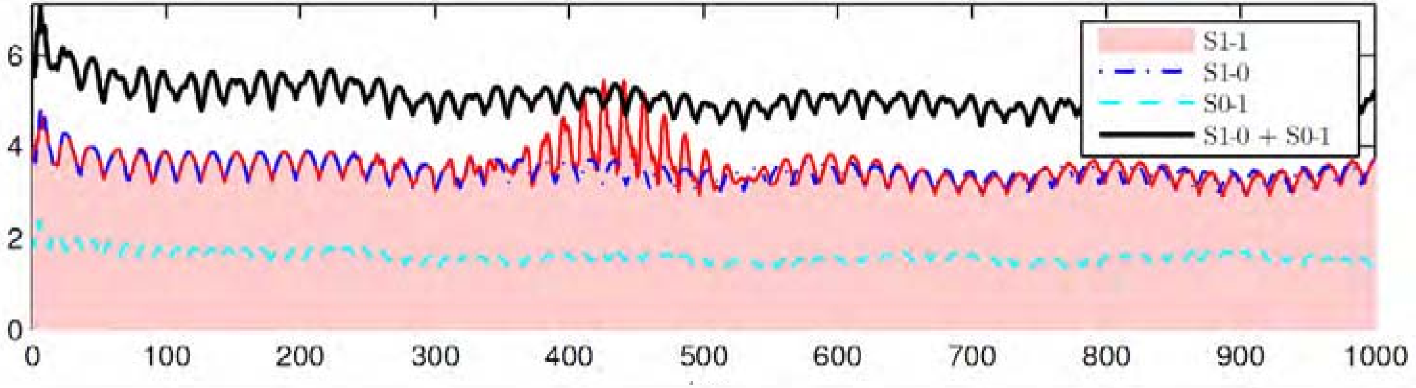
Integral mode amplitudes
for k = 0.05 rad/m



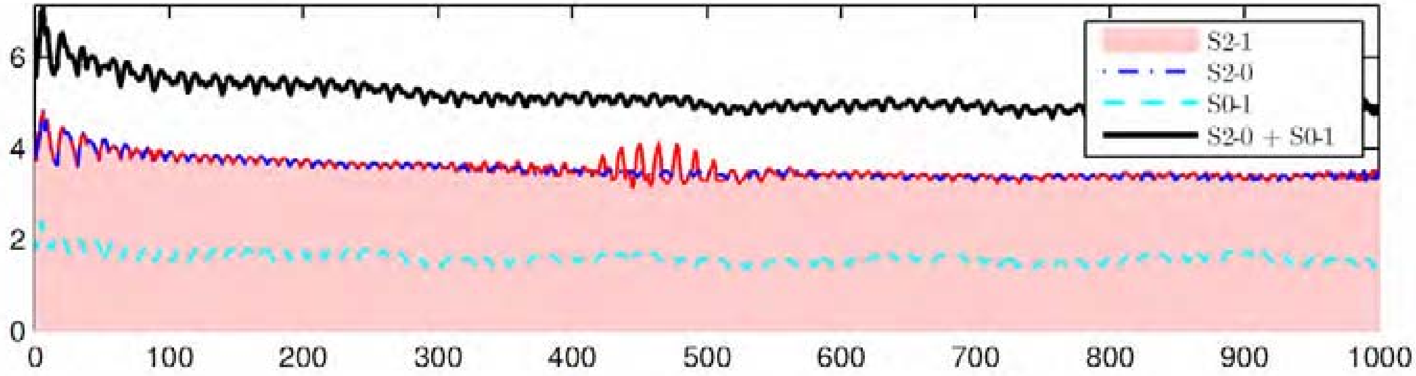
$$B_n(t)=\sqrt{\int b_n^2 dx} \qquad b_n(x,t)=\frac{\int \eta Y_n dy}{\int Y_n^2 dy}$$

Maximum displacements

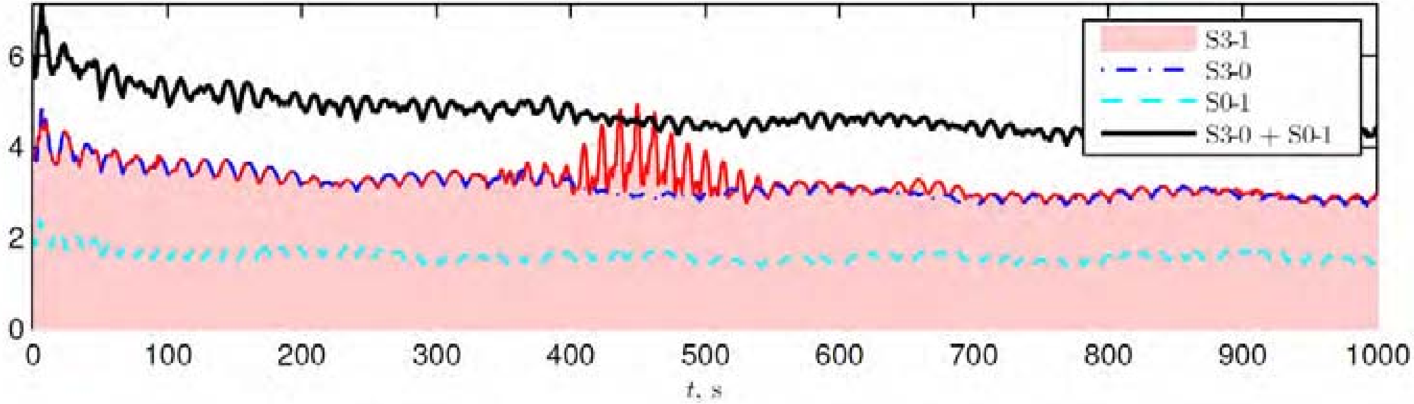
The maximum displacements as functions of time in the simulations of interacting solitons and in reference simulations of single solitons.



Two solitons of fundamental modes



Second mode and fundamental mode



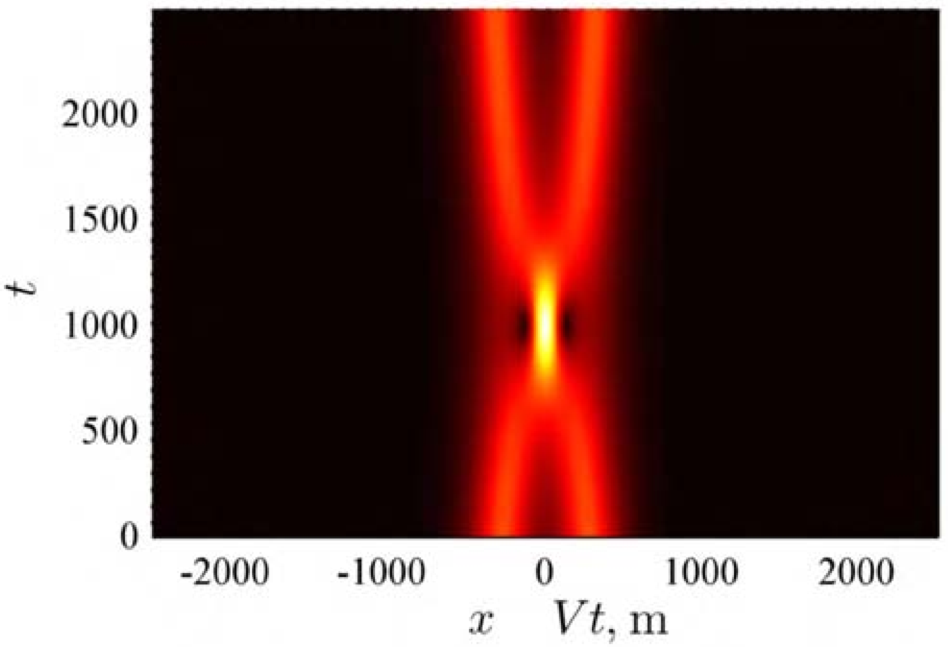
Third mode and fundamental mode

Degenerate 2-soliton NLSE solution

The solution [Peregrine, 1983; Akhmediev & Ankiewicz, 1993] describes interaction of two NLSE envelope solitons with identical amplitudes and velocities.

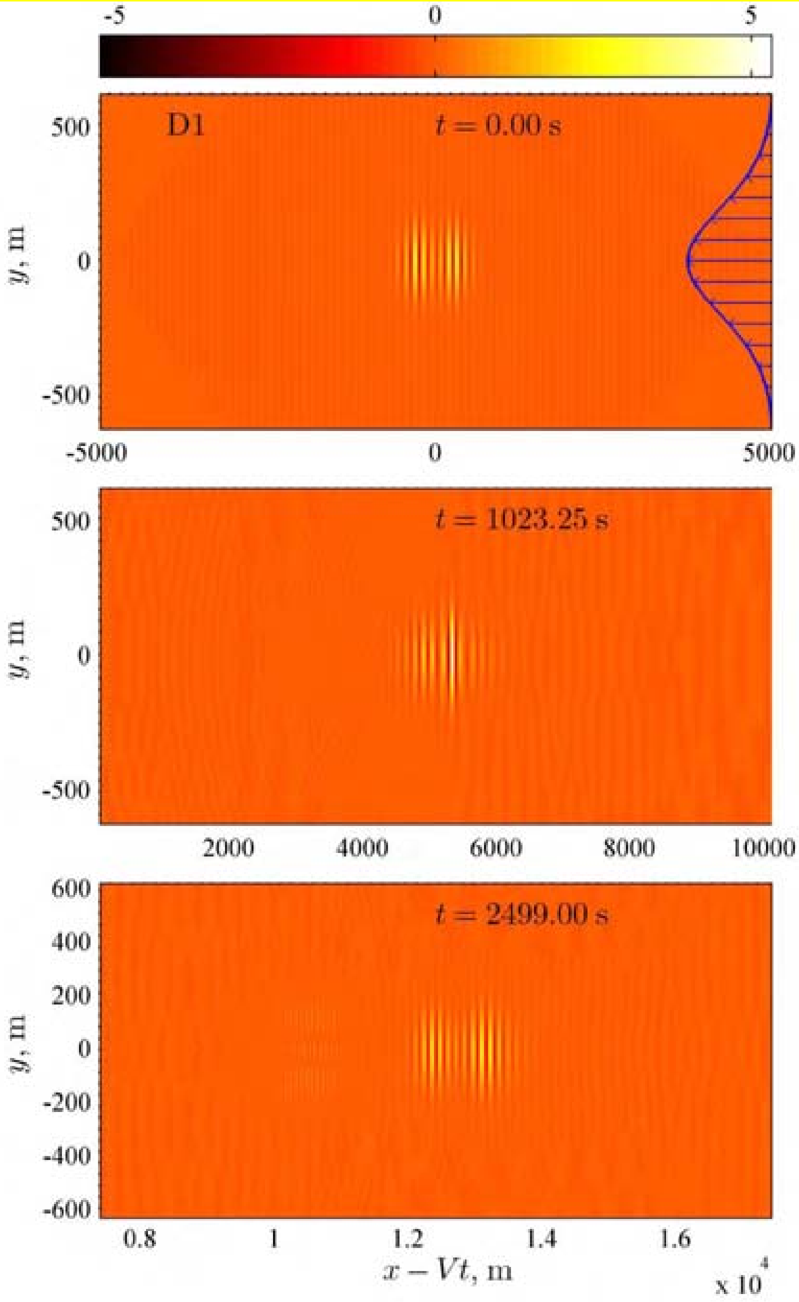
$$\psi_{\deg}(x,t) = 4\frac{a}{I}\frac{\xi\sinh\xi - \left(1+\frac{i}{2}k^2a^2\omega t\right)\cosh\xi}{\cosh 2\xi + 1 + 2\xi^2 + \frac{1}{2}k^4a^4\omega^2t^2}e^{i\frac{k^2a^2}{4}\omega t} \qquad \xi = \sqrt{2}k^2a(x-Vt)$$

The solution tends to two envelope solitons with amplitudes $A = a/I$ when $t \to \pm\infty$ and reaches at most twice larger amplitude $2A$ at $t = 0$ and $x = 0$.



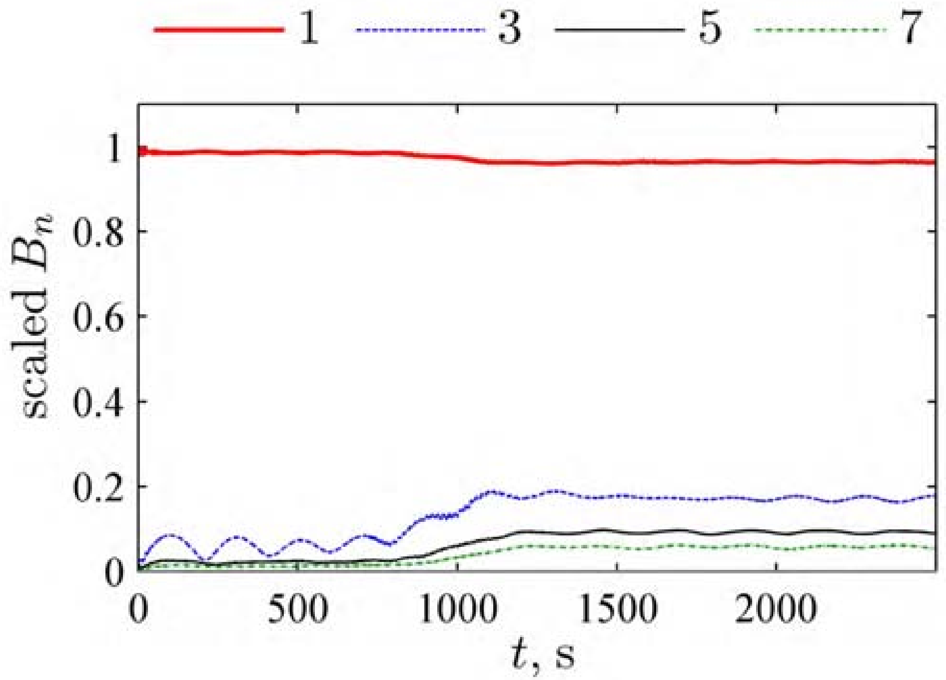
This is the longest possible interaction scenario for two solitons, which should reveal non-integrable features of the interaction.

Degenerate 2-soliton solution: simulation



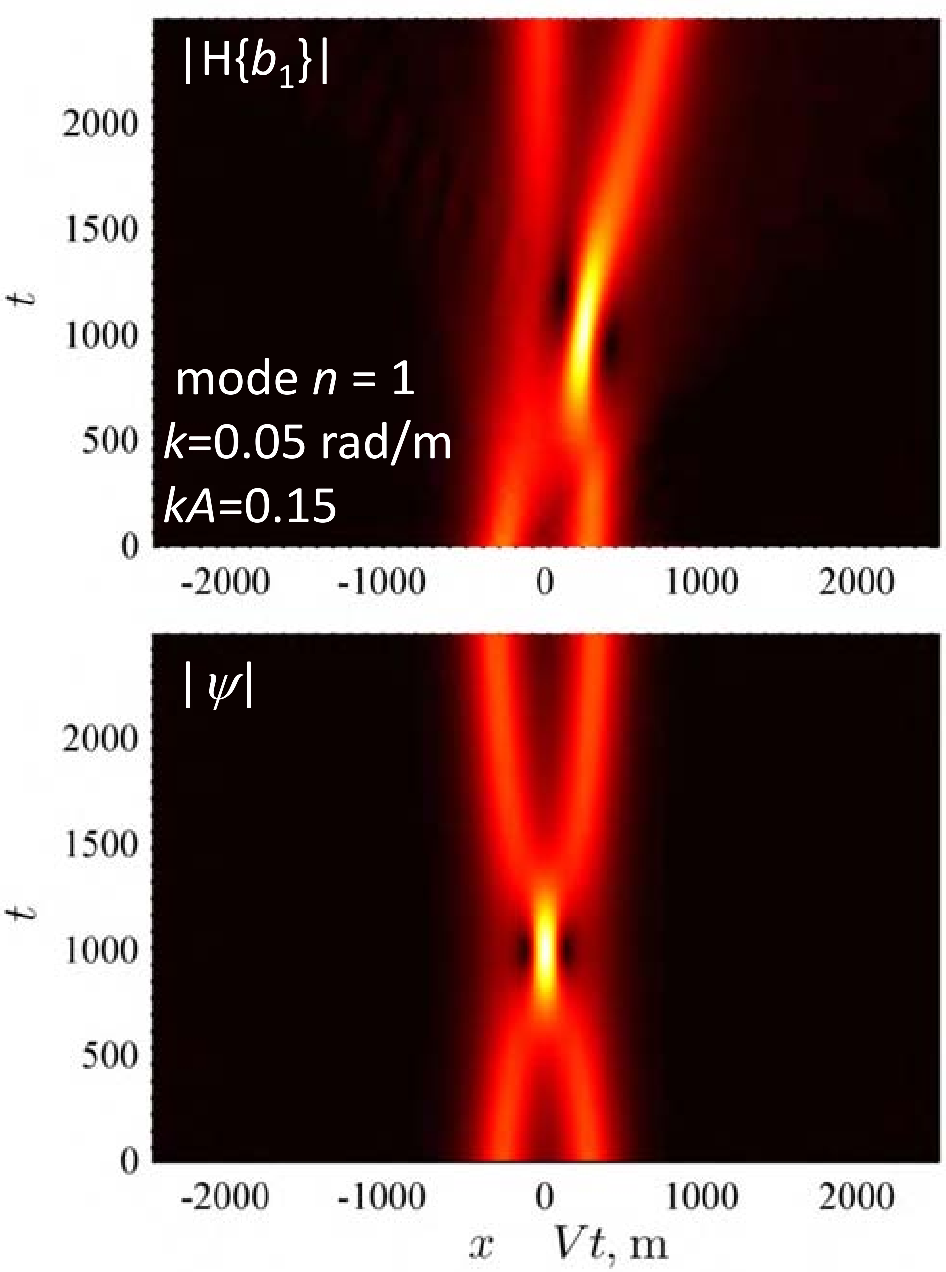
k = 0.05 rad/m & n = 1, kA = 0.15.
The wave period according to the solution of the SL problem: T = 10.3 s, V = 4.98 m/s.

Energy of the fundamental mode is partly transferred to other modes due to the imperfectly elastic collision



$$B_n(t) = \sqrt{\int b_n^2 dx} \qquad b_n(x,t) = \frac{\int \eta Y_n dy}{\int Y_n^2 dy}$$

Degenerate 2-soliton solution: simulation

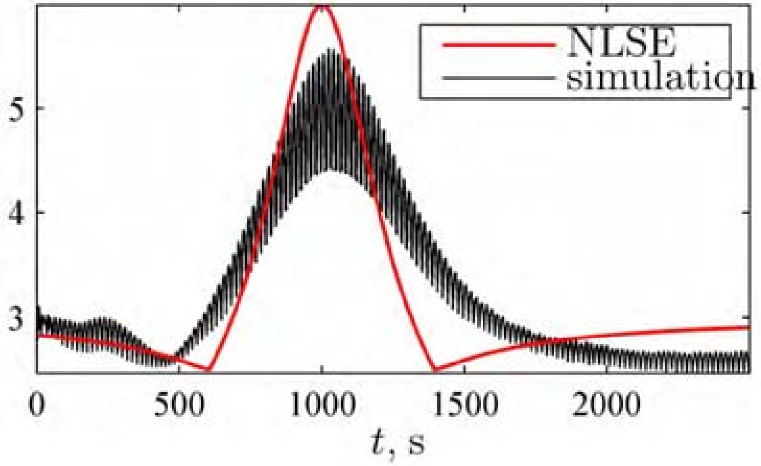


mode n = 1
k=0.05 rad/m
kA=0.15

The numerical simulations of primitive equations reproduces the stages of solitons' approach, generation of large-amplitude waves and then repulsing and consequent propagation as isolated pulses.

However, the symmetry of the interaction is lost. The leading soliton obtains greater amount of energy.

Maxima of $|\psi|$ and $|H\{b_1\}|$



Conclusions

❶ Trapped wave conditions represent a unique situation of surface water waves when the defocusing effect in the transversal direction is cancelled. Then the requirement on the small width of the angular spectrum for modulationally unstable waves is greatly relaxed, what can cause higher probability of rogue wave generation in ordinary sea conditions.

❷ The NLSE envelope solitons are structural elements of the coherent nonlinear wave dynamics which are involved in the modulational instability scenarios related to the generation of rogue waves. These long-lived groups exhibit own non-trivial dynamics; its understanding may be used for elaboration of warning criteria.

❸ We show that the dynamics of waves trapped by jet currents may be effectively described within the mode approach, which allows the formulation of simplified theories, including approximate analytic solutions. Modes can preserve energy for hundreds of wave periods.

❹ Envelopes solitons of trapped waves are stable structures, which can propagate for hundreds of wave periods with no noticeable loss. They interact to a great extent elastically and may cause rogue waves.

❺ A number of specific physical mechanisms related to adiabatically slow or rapid variation of the surrounding conditions may be suggested as possible mechanisms of wave amplification and generation of rogue waves.

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XXI НАУЧНАЯ ШКОЛА

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Нижний Новгород, 5 - 11 ноября

XXI Научная школа «НЕЛИНЕЙНЫЕ ВОЛНЫ – 2024»

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ТЕМАТИКА ШКОЛЫ

- Современные проблемы теории нелинейных колебаний и волн
- Нелинейные процессы в геофизике
- Модели климата и экосистем
- Нелинейные явления в космологии и астрофизике
- Нелинейная фотоника
- Нелинейные явления в физике плазмы и электронике
- Нелинейные процессы в биофизике и нейродинамике
- Нелинейная динамика квантовых систем

Прием заявок до **9 сентября 2024 г.**

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